Number Theory - Modular Arithmetic

Learn to solve this type of problems, not just this problem!

(1) Let integer a, b, and c satisfy $a+b+c=0$, show that $6 \mid a^{2017}+b^{2017}+c^{2017}$.

(2) Suppose two sequences $\{x_n\}$ and $\{y_n\}$ are defined as

$$x_1=1, x_2=1, x_{n+1}=x_n+2x_{n-1}$$

and

$$y_1=7, y_2=17, y_{n+1}=2y_n+3y_{n-1}$$

Show that no term in $\{x_n\}$ and $\{y_n\}$ will be equal.

(3) Find the last three digits of 7^{10000} and 7^{9999}.

(4) Find the last three digits of $9+9^2+9^3+\cdots+9^{2000}$.

(5) Find the remainder when $10^{10}+10^{100}+10^{1000}+\cdots+10^{\overbrace{10\cdots0}^{2017}}$ is divided by 7.

(6) Let $p>3$ be a prime and

$$\frac{a}{b}=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}$$

Show that a is a multiple of p.

(7) Solve the following equation in integers:

$$x_1^4+x_2^4+\cdots+x_{14}^4=9999$$

(8) Show that from any given m integers a_1, a_2, $\cdots$, a_m, it is always possible to select one or more integers so that their sum is a multiple of m.

(9) Show that among all seven-digit integers which are created by using all of 1, 2, $\cdots$, 7, none of them can be a multiple of another one.

(10) Find all powers of 2, such that after deleting its first digit, the new number is also a power of 2. For example, 32 is such a number because $32=2^5$ and $2=2^1$.

(11) Find the remainder when 30! is divided by 899.

(12) Show that for any positive integer n, there exist n consecutive integers each of which contains at least one square divisor greater than 1.