Art of Thinking Pre-assessment

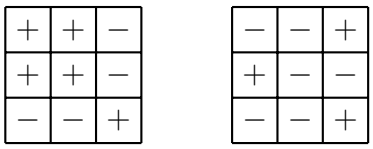
Competition math is not about complicated theorems and formulas.

(1) There are 13 squares of side length 1 positioned inside a circle of radius 2. Show that at least two of these squares have a common point.

(2) An ATM machine can only dispense two-dollar bills and five-dollar bills. Show that it is always capable of dispensing exactly n dollars when $n \geq 4$.

(3) Show that in a n-people party, at least two of them have met the same number of other guests before.

(4) Given two grids shown below, is it possible to transform (a) to (b) after a series of operations? In each operation, one can change all the signs in either one entire row or one entire column.



(a) (b)

(5) Given $n > 2$ points on a plane. Prove if any straight line passing two of these points must pass another point, then all these n points are collinear.

(6) Suppose the following system has one unique real number solution, find the value of m and solve this system.

$$\begin{cases} x^2 + y^2 &= z \\ x + y + z &= m \end{cases}$$

(7) There are three piles of stones, numbering 19, 8, and 9, respectively. You are allowed to choose two piles and transfer one stone from each of them to the third pile. Is it possible to make all piles all contain exactly 12 stones after several such operations?

(8) It is possible to use some $1 \times 2 \times 4$ blocks to construct a $6 \times 6 \times 6$ cubic?

(9) An executioner lines up 100 prisoners single file and puts a red or a blue hat on each prisoner's head. Every prisoner can see the hats of the people in front of him in the line - but not his own hat, nor those of anyone behind him. The executioner starts at the end (back) and asks the last prisoner the color of his hat. He must answer "red" or "blue." If he answers correctly, he is allowed to live. If he gives the wrong answer, he is killed instantly and silently. (While everyone hears the answer, no one knows whether an answer was right.) On the night before the line-up, the prisoners confer on strategy to help them. What should they do in order to save as many prisoners as possible.

(10) Find all pairs of positive integers (a, b) satisfying $a! + b! = a^b + b^a$.