Competition Algebra

Learn to solve this type of problems, not just this problem!

(1) Let a and b be the two roots of $x^2+x+1=0$. Evaluate a^2+b^2. Try to find at least three different solutions without solving the equation directly.

(2) Let x_1 and x_2 be two roots of $x^2-x-1=0$. Find the value of $2x_1^5+5x_2^3$.

(3) Let α and β be two real roots of $x^4+k=3x^2$ and also satisfy $\alpha+\beta=2$. Find the value of k.

(4) Compute the value of

$$\sqrt[3]{2+\frac{10}{3\sqrt{3}}}+\sqrt[3]{2-\frac{10}{3\sqrt{3}}}$$

and simplify

$$\sqrt[3]{2+\frac{10}{3\sqrt{3}}} \quad \text{and} \quad \sqrt[3]{2-\frac{10}{3\sqrt{3}}}$$

(5) Solve this equation $x^4+2x^3-3x^2-4x+3=0$.

(6) Solve this equation $(6x+7)^2(3x+4)(x+1)=6$.

(7) If all roots of the equation

$$x^4-16x^3+(81-2a)x^2+(16a-142)x+(a^2-21a+68)=0$$

are integers, find the value of a and solve this equation.

(8) Let x be a positive number. Denote by $\lfloor x\rfloor$ the integer part of x and by $\{x\}$ the decimal part of x. Find the sum of all positive numbers satisfying $5\{x\}+0.2\lfloor x\rfloor=25$.

(9) Is it possible to construct 12 geometric sequences to contain all the prime numbers between 1 and 100?

(10) Suppose α and β be two real roots of $x^2-px+q=0$ where p and $q\neq 0$ are two real numbers. Let sequence $\{a_n\}$ satisfies $a_1=p$, $a_2=p^2-q$, and $a_n=pa_{n-1}-qa_{n-2}$ for $n>2$.
   i) Express a_n using α and β.
   ii) If $p=1$ and $q=\frac{1}{4}$, find the sum of first n terms of $\{a_n\}$.

(11) If real number x satisfies $x^4-2x^3-7x^2+8x+12\leq 0$, find the maximum value of $\left| x+\frac{4}{x} \right|$.

(12) Solve the following system in integers:

$$\begin{cases} x_1+x_2+\cdots+x_n &= n \\ x_1^2+x_2^2+\cdots+x_n^2 &= n \\ \qquad\cdots & \\ x_1^n+x_2^n+\cdots+x_n^n &= n \end{cases}$$