# Assessment

# Power Calculation

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Assessment

# Power Calculation

## Practice 1

Compute the sums of the following expressions:

i) $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018$

ii) $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \cdots + \frac{1}{2016\times2017\times2018}$

## Practice 2

Let

$$f(r) = \sum_{j=2}^{2016} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2016^r}$$

Find the value of

$$\sum_{k=2}^{\infty} f(k)$$

## Practice 3

Find the values of the following nested radicals:

i) $\sqrt{5 + \sqrt{5^2 + \sqrt{5^4 + \sqrt{5^8 + \ldots}}}}$

ii) $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$

## Practice 4

Without using a calculator, explain why the following approximation holds:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} - \sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx 1$$

# Power Calculation

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## Practice 5

Find the length of the leading non-repeating block in the decimal expansion of $\frac{2017}{3\times 5^{2016}}$. For example the length of the leading non-repeating block of $\frac{1}{6} = 0.1\overline{6}$ is 1.

## Practice 6

Simplify:

i) $C_n^0 + 2C_n^1 + 4C_n^2 + \cdots + 2^n C_n^n$

ii) $C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n$

iii) $C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \cdots + \frac{1}{n+1}C_n^n$

iv) $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \cdots + (C_n^n)^2$

## Practice 7

Simplify:

i) $\sin\theta + 2\sin 2\theta + 3\sin 3\theta \cdots + n\sin n\theta$

ii) $\sin\theta + \frac{1}{2}\cdot\sin 2\theta + \frac{1}{4}\cdot\sin 3\theta + \cdots$

## Practice 8

Without using a calculator, find the value of $\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \frac{9\pi}{13}$.



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# Reference Solutions

# Power Calculation

## Practice 1

Compute the sums of the following expressions:

i) $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018$

ii) $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \cdots + \frac{1}{2016\times2017\times2018}$

*Note: as a practice, it is sufficient to give the answer as those in boxes.*

$$
\begin{aligned}
i) \quad & 1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018 \\
=& 3! \times \left( \frac{1 \times 2 \times 3}{3!} + \frac{2 \times 3 \times 4}{3!} + \cdots + \frac{2016 \times 2017 \times 2018}{3!} \right) \\
=& 3! \times \left( C_3^3 + C_4^3 + \cdots + C_{2018}^3 \right) \\
=& \boxed{3! \times C_{2019}^4} \\
=& 4141845698256
\end{aligned}
$$

$$
\begin{aligned}
ii) \quad & \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{2016 \times 2017 \times 2018} \\
=& \frac{1}{2} \times \left( \frac{1}{1 \times 2} - \frac{1}{2 \times 3} \right) + \frac{1}{2} \times \left( \frac{1}{2 \times 3} - \frac{1}{3 \times 4} \right) + \cdots + \frac{1}{2} \times \left( \frac{1}{2016 \times 2017} - \frac{1}{2017 \times 2018} \right) \\
=& \boxed{\frac{1}{2} \times \left( \frac{1}{1 \times 2} - \frac{1}{2017 \times 2018} \right)} \\
=& \frac{508788}{2035153}
\end{aligned}
$$

## Practice 2

Let

$$f(r) = \sum_{j=2}^{2016} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2016^r}$$

Find the value of

$$\sum_{k=2}^{\infty} f(k)$$

$$\begin{aligned}
\sum_{k=2}^{\infty} f(k) &= f(2)+f(3)+f(4)+\cdots \\
&= \tfrac{1}{2^2}+\tfrac{1}{3^2}+\cdots+\tfrac{1}{2016^2} \\
&+ \tfrac{1}{2^3}+\tfrac{1}{3^3}+\cdots+\tfrac{1}{2016^3} \\
&+ \tfrac{1}{2^4}+\tfrac{1}{3^4}+\cdots+\tfrac{1}{2016^4} \\
&+ \cdots \\
&= \left(\tfrac{1}{2^2}+\tfrac{1}{2^3}+\tfrac{1}{2^4}+\cdots\right) \\
&+ \left(\tfrac{1}{3^2}+\tfrac{1}{3^3}+\tfrac{1}{3^4}+\cdots\right) \\
&+ \cdots \\
&+ \left(\tfrac{1}{2016^2}+\tfrac{1}{2016^3}+\tfrac{1}{2016^4}+\cdots\right) \\
&= \tfrac{1}{2^2}\times\tfrac{1}{1-1/2}+\tfrac{1}{3^2}\times\tfrac{1}{1-1/3}+\cdots+\tfrac{1}{2016^2}\times\tfrac{1}{1-1/2016} \\
&= \tfrac{1}{2\times1}+\tfrac{1}{3\times2}+\cdots+\tfrac{1}{2016\times2015} \\
&= \left(\tfrac{1}{1}-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\cdots+\left(\tfrac{1}{2015}-\tfrac{1}{2016}\right) \\
&= \boxed{\frac{2015}{2016}}
\end{aligned}$$

## Practice 3

Find the values of the following nested radicals:

i) $\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\ldots}}}}$

ii) $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$

i) First let's compute $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$.

Let $S=\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}>0$, we find $S^2=1+S \implies S=\frac{\sqrt{5}+1}{2}$. Then

$$\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\ldots}}}}=\sqrt{5}\times\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}=\sqrt{5}\times\frac{\sqrt{5}+1}{2}=\frac{5+\sqrt{5}}{2}$$

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ii) Setting $n = 1$ and $x = 2$ in the Srinivasa Ramanujan identity below leads to the answer $\boxed{3}$.

$$x + n = \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + (x+2n)\sqrt{\cdots}}}}$$

## Practice 4

Without using a calculator, explain why the following approximation holds:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} - \sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx 1$$

The approximation holds because:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} \approx \sqrt{20 + \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}} = 5$$

$$\sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx \sqrt{20 - \sqrt{20 - \sqrt{20 - \sqrt{20 - \cdots}}}} = 4$$

## Practice 5

Find the length of the leading non-repeating block in the decimal expansion of $\frac{2017}{3\times 5^{2016}}$. For example the length of the leading non-repeating block of $\frac{1}{6} = 0.1\overline{6}$ is 1.

It is easy to see that the given expression can be decomposed as

$$\frac{2017}{3 \times 5^{2016}} = \frac{A}{3} + \frac{B}{5^{2016}}$$

where both $A$ and $B$ are two constants.

It is clear that $\frac{A}{3}$ is a repeating decimal from the tenths digit. Meanwhile, $\frac{B}{5^{2016}}$ is a decimal of 2016 digits to the right of the decimal point. Therefore, the length of the leading non-repeating block is 2016.

# Power Calculation

## Practice 6

Simplify:

i) $C_n^0 + 2C_n^1 + 4C_n^2 + \cdots + 2^n C_n^n$

ii) $C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n$

iii) $C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \cdots + \frac{1}{n+1}C_n^n$

iv) $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \cdots + (C_n^n)^2$

i) Setting $x = 2$ in the following identity gives the answer of $\boxed{3^n}$.

$$(1+x)^n = C_n^0 + C_n^1 x^2 + C_n^2 x^3 + \cdots + C_n^n x^n$$

ii) Let $S = C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n$. Then

$$\begin{array}{ccccccccc} S & = & 0 \times C_n^0 & + & 1 \times C_n^1 & + \cdots + & (n-1) \times C_n^{n-1} & + & n \times C_n^n \\ S & = & n \times C_n^n & + & (n-1) \times C_n^1 & + \cdots + & 1 \times C_n^{n-1} & + & 0 \times C_n^n \end{array}$$

Therefore

$$2 \cdot S = n \times \left( C_n^0 + C_n^1 + \cdots + C_n^{n-1} + C_n^n \right) = n \times 2^n \implies S = \boxed{n \times 2^{n-1}}$$

iii) By identity $\frac{1}{k+1}C_n^k = \frac{1}{n+1}C_{n+1}^{k+1}$, we have

$$\begin{aligned} & C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \cdots + \frac{1}{n+1}C_n^n \\ =& \frac{1}{n+1}C_{n+1}^1 + \frac{1}{n+1}C_{n+1}^2 + \cdots + \frac{1}{n+1}C_{n+1}^{n+1} \\ =& \frac{1}{n+1} \times \left( C_{n+1}^1 + C_{n+1}^2 + \cdots + C_{n+1}^{n+1} \right) \\ =& \frac{1}{n+1} \times (2^{n+1} - 1) \end{aligned}$$

iv) By Vandermonde identity:

$$\begin{aligned} & (C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \cdots + (C_n^n)^2 \\ =& C_n^0 C_n^n + C_n^1 C_n^{n-1} + C_n^2 C_n^{n-2} + \cdots + C_n^n C_n^0 \\ =& \boxed{C_{2n}^n} \end{aligned}$$

# Power Calculation

## Practice 7

Simplify:

i) $\sin\theta + 2\sin 2\theta + 3\sin 3\theta \cdots + n\sin n\theta$

ii) $\sin\theta + \frac{1}{2}\cdot\sin 2\theta + \frac{1}{4}\cdot\sin 3\theta + \cdots$

i) Let $S = \sin\theta + 2\sin 2\theta + 3\sin 3\theta \cdots + n\sin n\theta$. Then

$$\begin{aligned}2\cdot\cos\theta\cdot S &= \sin 2\theta + 2(\sin 3\theta + \sin\theta) + \cdots + n(\sin(n+1)\theta + \sin(n-1)\theta)\\ &= 2\cdot S + n\sin(n+1)\theta - (n+1)\sin n\theta\end{aligned}$$

Therefore

$$S = \boxed{\frac{(n+1)\sin n\theta - n\sin(n+1)\theta}{2(1-\cos\theta)}}$$

ii) Let $z = \cos\theta + i\sin\theta$, then

$$\begin{aligned}&\left(\cos\theta + \frac{1}{2}\cdot\cos 2\theta + \frac{1}{4}\cdot\cos 3\theta + \cdots\right) + i\left(\sin\theta + \frac{1}{2}\cdot 2\sin 2\theta + \frac{1}{4}\cdot\sin 3\theta + \cdots\right)\\ =&z + \frac{1}{2}\cdot z^2 + \frac{1}{4}\cdot z^3 + \cdots\\ =&z/(1 - \frac{1}{2}\cdot z)\\ =&\frac{2z}{2-z}\\ =&\frac{2(\cos\theta + i\sin\theta)}{(2-\cos\theta) - i\sin\theta}\\ =&\frac{2(\cos\theta + i\sin\theta)((2-\cos\theta) + i\sin\theta)}{((2-\cos\theta) - i\sin\theta)((2-\cos\theta) + i\sin\theta)}\\ =&\frac{(\cdots) + i(4\sin\theta)}{5 - 4\cos\theta}\end{aligned}$$

Therefore the answer is $\boxed{\dfrac{4\sin\theta}{5-4\cos\theta}}$.

# Power Calculation

## Practice 8

Without using a calculator, find the value of $\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\frac{9\pi}{13}$.

This problem requires the following identity:

$$\cos\frac{\pi}{2n+1}+\cos\frac{3\pi}{2n+1}+\cdots+\cos\frac{(2n-1)\pi}{2n+1}=\frac{1}{2}$$

Let $x=\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}$ and $y=\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{11\pi}{13}$. Then

$$x+y=\frac{1}{2}\quad\text{by the identity above}$$

and

$$\begin{aligned}
xy &= \left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}\right)\left(\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{11\pi}{13}\right)\\
&= -\frac{3}{2}\times\left(\cos\frac{\pi}{13}-\cos\frac{2\pi}{13}+\cos\frac{3\pi}{13}-\cos\frac{4\pi}{13}+\cos\frac{5\pi}{13}-\cos\frac{6\pi}{13}\right)\\
&= -\frac{3}{2}\times\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{11\pi}{13}\right)\\
&= -\frac{3}{2}\times\frac{1}{2}\\
&= -\frac{3}{4}
\end{aligned}$$

Therefore $x$ and $y$ are two roots of the following equation:

$$t^2-\frac{1}{2}\cdot t-\frac{3}{4}=0$$

Clearly $x>0$. Solving the above equation leads to $x=\boxed{\frac{1+\sqrt{13}}{4}}$.