Geometry

Basic Trigonometry in Geometry

Math All Star

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Basic Trigonometry in Geometry

Instructions

- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.* C_{100}^{50} *is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good practice?*
- *Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.*

Legends

Tips, additional information etc

Important theorem, conclusion to remember.

Addition questions for further study.

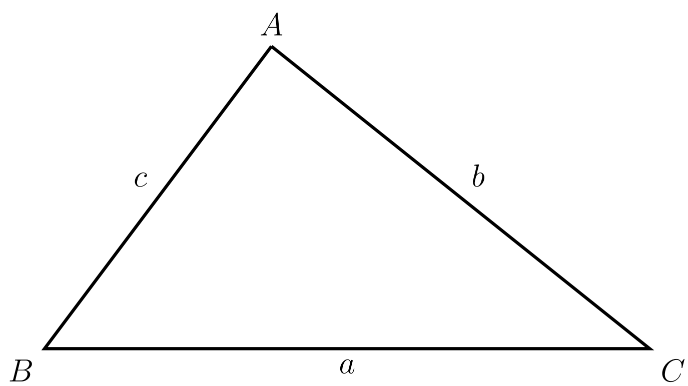
My Comments and Notes

Basic Trigonometry in Geometry

Unless otherwise noted, the following conventions will be followed in this practice:

In a given $\triangle ABC$:

- Uppercase letters A, B, and C represent measurements of internal angles
- Lowercase letters a, b, and c represent lengths of corresponding opposite sides



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Basic Trigonometry in Geometry

Practice 1

Let S be the area of $\triangle ABC$, show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$

Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= c^2 + a^2 - 2ca \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{cases}$$

Practice 4

(Circumradius) Let R be the circumradius of $\triangle ABC$, show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$

Basic Trigonometry in Geometry

Practice 5

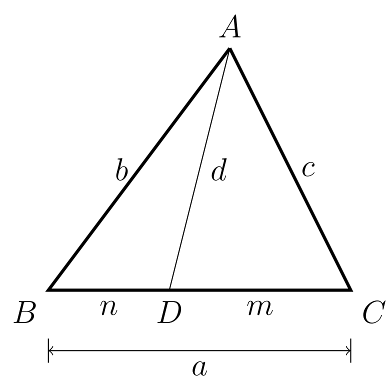
(Circumradius) Let S and R be the area and circumradius of $\triangle ABC$, respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$

Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

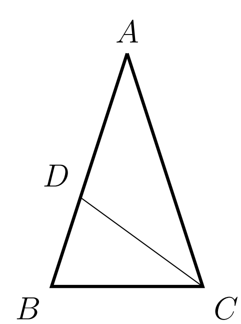
$$b^2m + c^2n = a(d^2 + mn)$$



Practice 7

($\sin 18°$) Utilizing the graph on the right to compute the value of $\sin 18°$.

$AB = AC$, $\angle A = 36°$, CD bisects $\angle ACB$

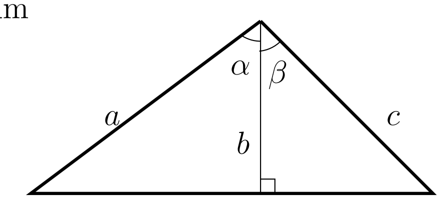


Basic Trigonometry in Geometry

Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



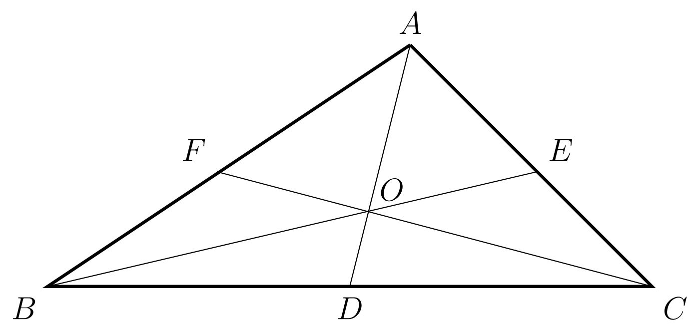
Practice 9

($\sin 15°$) Show that $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. Can you solve this problem using more than one method?

Practice 10

(Trigonometry Form of Civa's Theorem)
As shown on the right, show that:

$$\frac{\sin BAD}{\sin DAC}\cdot\frac{\sin ACF}{\sin FCB}\cdot\frac{\sin CBE}{\sin EBA} = 1$$



Reference Solutions

Basic Trigonometry in Geometry

Practice 1

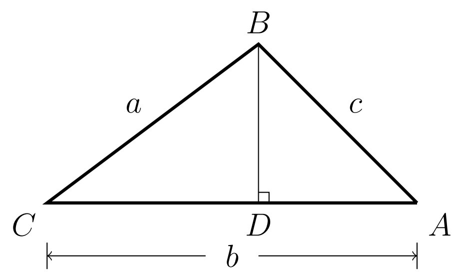
Let S be the area of $\triangle ABC$, show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$

Let's prove $S = \frac{1}{2} \cdot ab \sin C$ here. The other two relationships can be proved in a similar way.

Draw an altitude from B and let its foot on AC be D. Then we have (note that $\overline{BC} = a$ and $\overline{AC} = b$)

$$S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BD} = \frac{1}{2} \cdot \overline{AC} \cdot (\overline{BC} \cdot \sin C) = \frac{1}{2} \cdot ab \sin C$$



Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

From the previous practice, we know: $S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$, or

$$bc \sin A = ca \sin B = ab \sin C$$

Dividing every term with abc which is non-zero leading to the conclusion:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Basic Trigonometry in Geometry

Practice 3

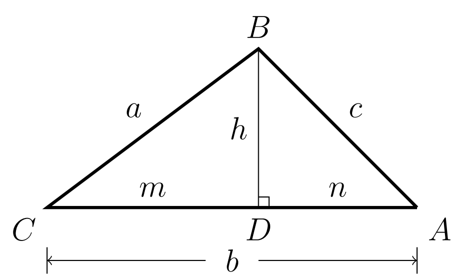
(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{cases}$$

Let's prove the first relationship here. The other two can be proved similarly. There are several different proofs. Here we present two of them.

Proof 1

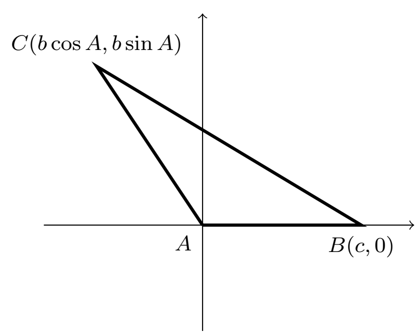
Draw an altitude from B and let its foot on AC be D. Note that $\overline{BC} = a$, $\overline{AC} = b$, and $\overline{AB} = c$, we have:



$$\begin{aligned} a^2 &= h^2 + m^2 = (c^2 - n^2) + (b-n)^2 \\ &= c^2 - n^2 + b^2 - 2bn + n^2 \\ &= b^2 + c^2 - 2b \cdot n \\ &= b^2 + c^2 - 2bc \cdot \sin A \end{aligned}$$

Proof 2

Let's put $\triangle ABC$ on a coordinate plane such that A is the origin and B is on the x-axis. It is then easy to see B's coordinate is $(c, 0)$ and C's coordinate is $(b\cos A, b\sin A)$. Hence, by the distance formula, we have:



$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A - 0)^2 \\ &= b^2\cos^2 A - 2bc \cdot \cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc \cdot \cos A \\ &= b^2 + c^2 - 2bc \cdot \cos A \end{aligned}$$

Tip: The combination of trigonometry and coordinate system provides a powerful way to transform a geometry problem to a straightforward computation.

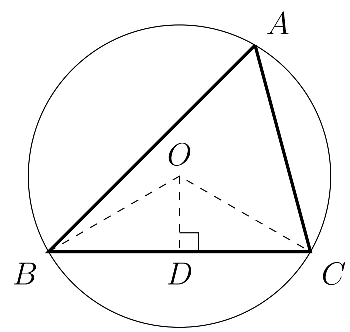
Basic Trigonometry in Geometry

Practice 4

(Circumradius) Let R be the circumradius of $\triangle ABC$, show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$

Let's prove $R = \dfrac{a}{2 \cdot \sin A}$ here. The other two relationships can be proved in a similar way.



Let O be the circumcircle of $\triangle ABC$. Connect OB, OC, and OD, where D is the middle point of BC.

O is the circumcenter $\implies OD \perp BC$ and OD bisects $\angle BOC$.

Meanwhile $\angle BOC = 2\angle A \implies \angle BOD = \angle A$

Now consider right $\triangle BOD$, we have $OB = R$, $BD = \dfrac{a}{2}$, and $\angle BOD = \angle A$. Therefore

$$BD = OB \cdot \angle BOD \implies \frac{a}{2} = R \sin A \implies R = \frac{a}{2 \cdot \sin A}$$

Practice 5

(Circumradius) Let S and R be the area and circumradius of $\triangle ABC$, respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$

By the previous practice, we know $R = \dfrac{a}{2 \sin A} = \dfrac{b}{2 \sin B}$. Hence:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot (2R \sin A)(2R \sin C) = 2R^2 \sin A \sin B \sin C$$

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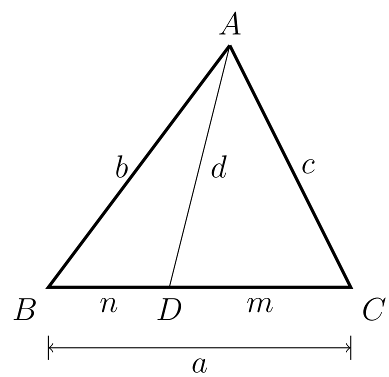
Basic Trigonometry in Geometry

Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$b^2m + c^2n = a(d^2 + mn)$$



By the Law of Cosine, we have:

$$\begin{cases} b^2 &= n^2 + d^2 - 2nd \cdot \cos\angle BDA \\ c^2 &= m^2 + d^2 - 2md \cdot \cos\angle CDA \end{cases}$$

Multiplying both sides of the 1^{st} equation by m, both sides of the 2^{nd} by n:

$$\begin{cases} b^2m &= n^2m + d^2m - 2mnd \cdot \cos\angle BDA \\ c^2n &= m^2n + d^2n - 2mnd \cdot \cos\angle CDA \end{cases}$$

Note that $\angle BDA + \angle CDA = \pi \implies \cos\angle BDA + \cos\angle CDA = 0$. Adding these two equations above give us:

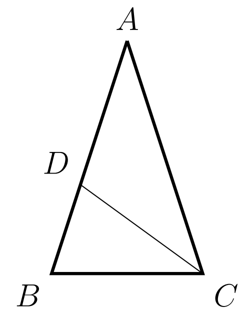
$$\begin{aligned} b^2m + c^2n &= (n^2m + m^2n) + (d^2m + d^2)n \\ b^2m + c^2n &= (n + m)mn + d^2(m + n) \\ b^2m + c^2n &= (n + m)(mn + d^2) \\ b^2m + c^2n &= a(d^2 + mn) \end{aligned}$$

Basic Trigonometry in Geometry

Practice 7

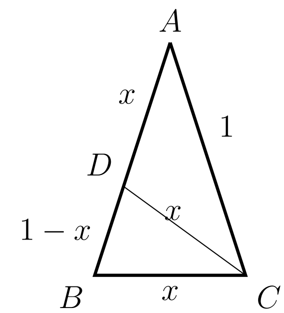
(sin 18°) Utilizing the graph on the right to compute the value of $\sin 18°$.

$AB = AC$, $\angle A = 36°$, CD bisects $\angle ACB$



It is easy to check that $\angle DCA = 36° = \angle A$, and $\angle BDC = 72° = \angle B$.

Hence we have $AD = DC = CB$. Without loss of generality, let $AC = AB = 1$, $AD = DC = CB = x$, and $DB = 1 - x$.



By the angle bisector theorem, we have

$$\frac{AC}{AD} = \frac{CB}{BD} \implies \frac{1}{x} = \frac{x}{1-x}$$

Solving the above equation and discarding the negative value give us $x = \frac{\sqrt{5}-1}{2}$.

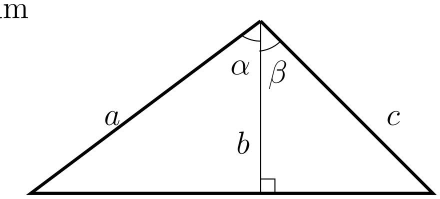
Because $\angle A = 36°$ and $AB = AC$, we have

$$\sin 18° = \frac{\frac{x}{2}}{1} = \frac{\sqrt{5}-1}{4}$$

Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



We can employ the area method here. Clearly the area of the bigger triangle equals the sum of

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those of the two smaller ones. Therefore, we have:

$$\begin{aligned}\frac{1}{2}ac\cdot\sin(\alpha+\beta) &= \frac{1}{2}ab\cdot\sin\alpha+\frac{1}{2}bc\cdot\sin\beta\\ \sin(\alpha+\beta) &= \frac{b}{c}\cdot\sin\alpha+\frac{b}{a}\cdot\sin\beta\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$

Practice 9

($\sin 15°$) Show that $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. Can you solve this problem using more than one method?

Tip: This problem can be solved using the angle bisector theorem, just as we did to compute $\sin 18°$ *earlier. Meanwhile, it can also be solved using the Sum of Sine formula we just derived.*

Tip: We will show the 2^{nd} *approach. It involves some techniques to simplify computation. You are encouraged to use the angle bisector theorem to solve this problem yourself.*

Setting $\alpha=\beta$ in the sum of sine formula leads to the double angle formula:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Further setting $\alpha = 15°$ yields:

$$\sin 30° = 2\cdot\sin 15°\cos 15°$$

Let $x=\sin 15°$, we have

$$\begin{aligned}\frac{1}{2} &= 2x\sqrt{1-x^2}\\ 1 &= 4x\sqrt{1-x^2}\\ 1 &= 16x^2(1-x^2)\\ 16x^4-16x^2+1 &= 0\\ (4x^2-2)^2 &= 3\\ x^2 &= \frac{2\pm\sqrt{3}}{4}\end{aligned}$$

Basic Trigonometry in Geometry

Because $0 < \sin 15° < \sin 30° = \frac{1}{2} \implies x^2 < \frac{1}{4}$, it must hold that

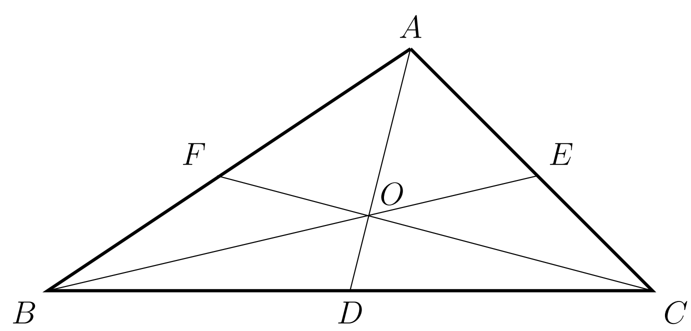
$$x^2 = \frac{2-\sqrt{3}}{4} \implies x = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Tip: The technique used to simplify nested radical expression in the last step is discussed in the Power Calculation lecture and practice.

Practice 10

(Trigonometry Form of Civa's Theorem)
As shown on the right, show that:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = 1$$



Tip: When cevians and ratios are involved, the area method is always a good candidate to consider.

We have the following relationship:

$$\frac{S_{\triangle ABO}}{S_{\triangle AOC}} = \frac{\frac{1}{2}\cdot AB \cdot AO \cdot \sin \triangle BAD}{\frac{1}{2}\cdot AO \cdot AC \cdot \sin \triangle DAC} = \frac{AB \cdot \sin \angle BAD}{AC \cdot \sin \angle DAC} \implies \frac{\sin \angle BAD}{\sin \angle DAC} = \frac{S_{\triangle ABO}\cdot AC}{S_{\triangle OAC}\cdot AB}$$

Similarly, we have:

$$\frac{\sin \angle ACF}{\sin \angle FCB} = \frac{S_{\triangle ACO}\cdot BC}{S_{\triangle OCB}\cdot AC}$$

$$\frac{\sin \angle CBE}{\sin \angle EBA} = \frac{S_{\triangle CBO}\cdot AB}{S_{\triangle OBA}\cdot BC}$$

Multiplying these three equations gives us:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = \frac{S_{\triangle ABO}\cdot AC}{S_{\triangle OAC}\cdot AB} \cdot \frac{S_{\triangle ACO}\cdot BC}{S_{\triangle OCB}\cdot AC} \cdot \frac{S_{\triangle CBO}\cdot AB}{S_{\triangle OBA}\cdot BC} = 1$$