Geometry

Basic Trigonometry in Geometry



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Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g. C_{100}^{50} is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good practice?
- Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.

Legends

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- Tips, additional information etc
- Important theorem, conclusion to remember.
 - Addition questions for further study.

My Comments and Notes

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Unless otherwise noted, the following conventions will be followed in this practice: In a given $\triangle ABC$:

- Uppercase letters A, B, and C represent measurements of internal angles
- Lowercase letters a, b, and c represent lengths of corresponding opposite sides



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Practice 1

Let S be the area of $\triangle ABC$, show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$

Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$

Practice 4

(Circumradius) Let R be the circumradius of $\triangle ABC$, show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$

Geometry Basic Trigonometry in Geometry Practice 5 (Circumradius) Let S and R be the area and circumradius of $\triangle ABC$, respectively, show that $S = 2R^2 \sin A \sin B \sin C$ Practice 6 A (Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show

that the following relationship holds





Practice 7

(sin 18°) Utilizing the graph on the right to compute the value of $\sin 18^{\circ}$.

$$AB = AC, \ \angle A = 36^{\circ}, \ CD \text{ bisects } \ \angle ACB$$





Reference Solutions

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Practice 1

Let S be the area of $\triangle ABC$, show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$

Let's prove $S = \frac{1}{2} \cdot ab \sin C$ here. The other two relationships can be proved in a similar way.

Draw an altitude from B and let its foot on AC be D. Then we have (note that $\overline{BC} = a$ and $\overline{AC} = b$)

$$S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BD} = \frac{1}{2} \cdot \overline{AC} \cdot (\overline{BC} \cdot \sin C) = \frac{1}{2} \cdot ab \sin C$$



Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

 \checkmark

From the previous practice, we know: $S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$, or

$$bc\sin A = ca\sin B = ab\sin C$$

Dividing every term with *abc* which is non-zero leading to the conclusion:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

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Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$

 \checkmark

Let's prove the first relationship here. The other two can be proved similarly. There are several different proofs. Here we present two of them.

Proof 1

Draw an altitude from B and let its foot on AC be D. Note that $\overline{BC} = a$, $\overline{AC} = b$, and $\overline{AB} = c$, we have:

$$a^{2} = h^{2} + m^{2} = (c^{2} - n^{2}) + (b - n)^{2}$$

= $c^{2} - n^{2} + b^{2} - 2bn + n^{2}$
= $b^{2} + c^{2} - 2b \cdot n$
= $b^{2} + c^{2} - 2bc \cdot \sin A$



$\underline{\text{Proof } 2}$

Let's put $\triangle ABC$ on a coordinate plane such that A is the origin and B is on the *x*-axis. It is then easy to see B's coordinate is (c, 0) and C's coordinate is $(b \cos A, b \sin A)$. Hence, by the distance formula, we have:

$$a^{2} = (b \cos A - c)^{2} + (b \sin A - 0)^{2}$$

= $b^{2} \cos^{2} A - 2bc \cdot \cos A + c^{2} + b^{2} \sin^{2} A$
= $b^{2} (\cos^{2} A + \sin^{2} A) + c^{2} - 2bc \cdot \cos A$
= $b^{2} + c^{2} - 2bc \cdot \cos A$



Tip: The combination of trigonometry and coordinate system provides a powerful way to transform a geometry problem to a straightforward computation.

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Practice 4

(Circumradius) Let R be the circumradius of $\triangle ABC$, show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$

Let's prove $R = \frac{a}{2 \cdot \sin A}$ here. The other two relationships can be proved in a similar way.

Let O be the circumcircle of $\triangle ABC$. Connect OB, OC, and OD, where D is the middle point of BC.

O is the circumcenter $\implies OD \perp BC$ and OD bisects $\angle BOC$.

Meanwhile $\angle BOC = 2 \angle A \implies \angle BOD = \angle A$



Now consider right $\triangle BOD$, we have OB = R, $BD = \frac{a}{2}$, and $\angle BOD = \angle A$. Therefore

$$BD = OB \cdot \angle BOD \implies \frac{a}{2} = R \sin A \implies R = \frac{a}{2 \cdot \sin A}$$

Practice 5

(Circumradius) Let S and R be the area and circumradius of $\triangle ABC$, respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$

By the previous practice, we know $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B}$. Hence:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot (2R \sin A)(2R \sin C) = 2R^2 \sin A \sin B \sin C$$

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Geometry Basic Trigonometry in Geometry Practice 6 (Steward Theorem), Given a triangle as

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$b^2m + c^2n = a(d^2 + mn)$$



By the Law of Cosine, we have:

$$\begin{cases} b^2 = n^2 + d^2 - 2nd \cdot \cos \angle BDA \\ c^2 = m^2 + d^2 - 2md \cdot \cos \angle CDA \end{cases}$$

Multiplying both sides of the 1^{st} equation by m, both sides of the 2^{nd} by n:

$$\left\{ \begin{array}{ll} b^2m &= n^2m + d^2m - 2mnd \cdot \cos \angle BDA \\ c^2n &= m^2n + d^2n - 2mnd \cdot \cos \angle CDA \end{array} \right.$$

Note that $\angle BDA + \angle CDA = \pi \implies \cos \angle BDA + \cos \angle CDA = 0$. Adding these two equations above give us:

$$b^{2}m + c^{2}n = (n^{2}m + m^{2}n) + (d^{2}m + d^{2})n$$

$$b^{2}m + c^{2}n = (n + m)mn + d^{2}(m + n)$$

$$b^{2}m + c^{2}n = (n + m)(mn + d^{2})$$

$$b^{2}m + c^{2}n = a(d^{2} + mn)$$

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(sin 18°) Utilizing the graph on the right to compute the value of $\sin 18^{\circ}$.

 $AB = AC, \angle A = 36^{\circ}, CD$ bisects $\angle ACB$



Α

It is easy to check that $\angle DCA = 36^\circ = \angle A$, and $\angle BDC = 72^\circ = \angle B$.

Hence we have AD = DC = CB. Without loss of generality, let AC = AB = 1, AD = DC = CB = x, and DB = 1 - x.

By the angle bisector theorem, we have

$$\frac{AC}{AD} = \frac{CB}{BD} \implies \frac{1}{x} = \frac{x}{1-x}$$



Solving the above equation and discarding the negative value give us $x = \frac{\sqrt{5}-1}{2}$.

Because $\angle A = 36^{\circ}$ and AB = AC, we have

$$\sin 18^{\circ} = \frac{\frac{x}{2}}{1} = \frac{\sqrt{5} - 1}{4}$$

Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



We can employ the area method here. Clearly the area of the bigger triangle equals the sum of

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those of the two smaller ones. Therefore, we have:

$$\frac{1}{2}ac \cdot \sin(\alpha + \beta) = \frac{1}{2}ab \cdot \sin\alpha + \frac{1}{2}bc \cdot \sin\beta$$
$$\sin(\alpha + \beta) = \frac{b}{c} \cdot \sin\alpha + \frac{b}{a} \cdot \sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Practice 9

(sin 15°) Show that sin 15° = $\frac{\sqrt{6} - \sqrt{2}}{4}$. Can you solve this problem using more than one method?

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- Tip: This problem can be solved using the angle bisector theorem, just as we did to compute sin 18° earlier. Meanwhile, it can also be solved using the Sum of Sine formula we just derived.
- **(i)** Tip: We will show the 2nd approach. It involves some techniques to simplify computation. You are encouraged to use the angle bisector theorem to solve this problem yourself.

Setting $\alpha = \beta$ in the sum of sine formula leads to the double angle formula:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Further setting $\alpha = 15^{\circ}$ yields:

$$\sin 30^\circ = 2 \cdot \sin 15^\circ \cos 15^\circ$$

Let $x = \sin 15^\circ$, we have

$$\frac{1}{2} = 2x\sqrt{1-x^2}$$

$$1 = 4x\sqrt{1-x^2}$$

$$1 = 16x^2(1-x^2)$$

$$16x^4 - 16x^2 + 1 = 0$$

$$(4x^2 - 2)^2 = 3$$

$$x^2 = \frac{2 \pm \sqrt{3}}{4}$$

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Because
$$0 < \sin 15^\circ < \sin 30^\circ = \frac{1}{2} \implies x^2 < \frac{1}{4}$$
, it must hold that

$$x^{2} = \frac{2-\sqrt{3}}{4} \implies x = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Tip: The technique used to simplify nested radical expression in the last step is discussed in the Power Calculation lecture and practice.

Practice 10



Tip: When cevians and ratios are involved, the area method is always a good candidate to consider.

We have the following relationship:

$$\frac{S_{\triangle ABO}}{S_{\triangle AOC}} = \frac{\frac{1}{2} \cdot AB \cdot AO \cdot \sin \triangle BAD}{\frac{1}{2} \cdot AO \cdot AC \cdot \sin \triangle DAC} = \frac{AB \cdot \sin \angle BAD}{AC \cdot \sin \angle DAC} \implies \frac{\sin \angle BAD}{\sin \angle DAC} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle OAC} \cdot AB}$$

Similarly, we have:

$$\frac{\sin \angle ACF}{\sin \angle FCB} = \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC}$$
$$\frac{\sin \angle CBE}{\sin \angle EBA} = \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC}$$

Multiplying these three equations gives us:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle OAC} \cdot AB} \cdot \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC} \cdot \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC} = 1$$