## Geometry

## Basic Trigonometry in Geometry


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## Basic Trigonometry in Geometry



## Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g. $C_{100}^{50}$ is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good practice?
- Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.


## Legends

(i) Tips, additional information etc
(2) Important theorem, conclusion to remember.
(1) Addition questions for further study.

## My Comments and Notes

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Unless otherwise noted, the following conventions will be followed in this practice:
In a given $\triangle A B C$ :

- Uppercase letters $A, B$, and $C$ represent measurements of internal angles
- Lowercase letters $a, b$, and $c$ represent lengths of corresponding opposite sides



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## Practice 1

Let $S$ be the area of $\triangle A B C$, show that:

$$
S=\frac{1}{2} \cdot a b \sin C=\frac{1}{2} \cdot b c \sin A=\frac{1}{2} \cdot c a \sin B
$$

## Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$
\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}
$$

## Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$
\left\{\begin{array}{l}
a^{2}=b^{2}+c^{2}-2 b c \cos A \\
b^{2}=c^{2}+a^{2}-2 c a \cos B \\
c^{2}=a^{2}+b^{2}-2 a b \cos C
\end{array}\right.
$$

Practice 4
(Circumradius) Let $R$ be the circumradius of $\triangle A B C$, show that

$$
R=\frac{a}{2 \cdot \sin A}=\frac{b}{2 \cdot \sin B}=\frac{c}{2 \cdot \sin C}
$$

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## Practice 5

(Circumradius) Let $S$ and $R$ be the area and circumradius of $\triangle A B C$, respectively, show that

$$
S=2 R^{2} \sin A \sin B \sin C
$$

## Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$
b^{2} m+c^{2} n=a\left(d^{2}+m n\right)
$$



## Practice 7

$\left(\sin 18^{\circ}\right)$ Utilizing the graph on the right to compute the value of $\sin 18^{\circ}$.
$A B=A C, \angle A=36^{\circ}, C D$ bisects $\angle A C B$


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## Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$
\sin (\alpha+\beta)=\sin \alpha \cos \beta+\cos \alpha \sin \beta
$$



## Practice 9

$\left(\sin 15^{\circ}\right)$ Show that $\sin 15^{\circ}=\frac{\sqrt{6}-\sqrt{2}}{4}$. Can you solve this problem using more than one method?

## Practice 10

(Trigonometry Form of Civa's Theorem)
As shown on the right, show that:

$$
\frac{\sin B A D}{\sin D A C} \cdot \frac{\sin A C F}{\sin F C B} \cdot \frac{\sin C B E}{\sin E B A}=1
$$



## Reference Solutions

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## Practice 1

Let $S$ be the area of $\triangle A B C$, show that:

$$
S=\frac{1}{2} \cdot a b \sin C=\frac{1}{2} \cdot b c \sin A=\frac{1}{2} \cdot c a \sin B
$$

Let's prove $S=\frac{1}{2} \cdot a b \sin C$ here. The other two relationships can be proved in a similar way.

Draw an altitude from $B$ and let its foot on $A C$ be $D$. Then we have (note that $\overline{B C}=a$ and $\overline{A C}=b$ )

$$
S=\frac{1}{2} \cdot \overline{A C} \cdot \overline{B D}=\frac{1}{2} \cdot \overline{A C} \cdot(\overline{B C} \cdot \sin C)=\frac{1}{2} \cdot a b \sin C
$$



## Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$
\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}
$$

From the previous practice, we know: $S=\frac{1}{2} \cdot a b \sin C=\frac{1}{2} \cdot b c \sin A=\frac{1}{2} \cdot c a \sin B$, or

$$
b c \sin A=c a \sin B=a b \sin C
$$

Dividing every term with $a b c$ which is non-zero leading to the conclusion:

$$
\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}
$$

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## - Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$
\left\{\begin{array}{l}
a^{2}=b^{2}+c^{2}-2 b c \cos A \\
b^{2}=c^{2}+a^{2}-2 c a \cos B \\
c^{2}=a^{2}+b^{2}-2 a b \cos C
\end{array}\right.
$$

Let's prove the first relationship here. The other two can be proved similarly. There are several different proofs. Here we present two of them.

## Proof 1

Draw an altitude from $B$ and let its foot on $A C$ be $D$. Note that $\overline{B C}=a, \overline{A C}=b$, and $\overline{A B}=c$, we have:

$$
\begin{aligned}
a^{2} & =h^{2}+m^{2}=\left(c^{2}-n^{2}\right)+(b-n)^{2} \\
& =c^{2}-n^{2}+b^{2}-2 b n+n^{2} \\
& =b^{2}+c^{2}-2 b \cdot n \\
& =b^{2}+c^{2}-2 b c \cdot \sin A
\end{aligned}
$$



## Proof 2

Let's put $\triangle A B C$ on a coordinate plane such that $A$ is the origin and $B$ is on the $x$-axis. It is then easy to see $B$ 's coordinate is $(c, 0)$ and $C$ 's coordinate is $(b \cos A, b \sin A)$. Hence, by the distance formula, we have:

$$
\begin{aligned}
a^{2} & =(b \cos A-c)^{2}+(b \sin A-0)^{2} \\
& =b^{2} \cos ^{2} A-2 b c \cdot \cos A+c^{2}+b^{2} \sin ^{2} A \\
& =b^{2}\left(\cos ^{2} A+\sin ^{2} A\right)+c^{2}-2 b c \cdot \cos A \\
& =b^{2}+c^{2}-2 b c \cdot \cos A
\end{aligned}
$$


(i) Tip: The combination of trigonometry and coordinate system provides a powerful way to transform a geometry problem to a straightforward computation.

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## Practice 4

(Circumradius) Let $R$ be the circumradius of $\triangle A B C$, show that

$$
R=\frac{a}{2 \cdot \sin A}=\frac{b}{2 \cdot \sin B}=\frac{c}{2 \cdot \sin C}
$$

Let's prove $R=\frac{a}{2 \cdot \sin A}$ here. The other two relationships can be proved in a similar way.

Let $O$ be the circumcircle of $\triangle A B C$. Connect $O B, O C$, and $O D$, where $D$ is the middle point of $B C$.
$O$ is the circumcenter $\Longrightarrow O D \perp B C$ and $O D$ bisects $\angle B O C$.
Meanwhile $\angle B O C=2 \angle A \Longrightarrow \angle B O D=\angle A$


Now consider right $\triangle B O D$, we have $O B=R, B D=\frac{a}{2}$, and $\angle B O D=\angle A$. Therefore

$$
B D=O B \cdot \angle B O D \Longrightarrow \frac{a}{2}=R \sin A \Longrightarrow R=\frac{a}{2 \cdot \sin A}
$$

## Practice 5

(Circumradius) Let $S$ and $R$ be the area and circumradius of $\triangle A B C$, respectively, show that

$$
S=2 R^{2} \sin A \sin B \sin C
$$

By the previous practice, we know $R=\frac{a}{2 \sin A}=\frac{b}{2 \sin B}$. Hence:

$$
S=\frac{1}{2} \cdot a b \sin C=\frac{1}{2} \cdot(2 R \sin A)(2 R \sin C)=2 R^{2} \sin A \sin B \sin C
$$

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## Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$
b^{2} m+c^{2} n=a\left(d^{2}+m n\right)
$$



By the Law of Cosine, we have:

$$
\left\{\begin{array}{l}
b^{2}=n^{2}+d^{2}-2 n d \cdot \cos \angle B D A \\
c^{2}=m^{2}+d^{2}-2 m d \cdot \cos \angle C D A
\end{array}\right.
$$

Multiplying both sides of the $1^{\text {st }}$ equation by $m$, both sides of the $2^{\text {nd }}$ by $n$ :

$$
\left\{\begin{aligned}
b^{2} m & =n^{2} m+d^{2} m-2 m n d \cdot \cos \angle B D A \\
c^{2} n & =m^{2} n+d^{2} n-2 m n d \cdot \cos \angle C D A
\end{aligned}\right.
$$

Note that $\angle B D A+\angle C D A=\pi \Longrightarrow \cos \angle B D A+\cos \angle C D A=0$. Adding these two equations above give us:

$$
\begin{aligned}
b^{2} m+c^{2} n & =\left(n^{2} m+m^{2} n\right)+\left(d^{2} m+d^{2}\right) n \\
b^{2} m+c^{2} n & =(n+m) m n+d^{2}(m+n) \\
b^{2} m+c^{2} n & =(n+m)\left(m n+d^{2}\right) \\
b^{2} m+c^{2} n & =a\left(d^{2}+m n\right)
\end{aligned}
$$

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## - Practice 7

$\left(\sin 18^{\circ}\right)$ Utilizing the graph on the right to compute the value of $\sin 18^{\circ}$.

$$
A B=A C, \angle A=36^{\circ}, C D \text { bisects } \angle A C B
$$



It is easy to check that $\angle D C A=36^{\circ}=\angle A$, and $\angle B D C=72^{\circ}=\angle B$.
Hence we have $A D=D C=C B$. Without loss of generality, let $A C=A B=1, A D=D C=C B=x$, and $D B=1-x$.

By the angle bisector theorem, we have

$$
\frac{A C}{A D}=\frac{C B}{B D} \Longrightarrow \frac{1}{x}=\frac{x}{1-x}
$$



Solving the above equation and discarding the negative value give us $x=\frac{\sqrt{5}-1}{2}$.
Because $\angle A=36^{\circ}$ and $A B=A C$, we have

$$
\sin 18^{\circ}=\frac{\frac{x}{2}}{1}=\frac{\sqrt{5}-1}{4}
$$

## Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$
\sin (\alpha+\beta)=\sin \alpha \cos \beta+\cos \alpha \sin \beta
$$



We can employ the area method here. Clearly the area of the bigger triangle equals the sum of

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those of the two smaller ones. Therefore, we have:

$$
\begin{aligned}
\frac{1}{2} a c \cdot \sin (\alpha+\beta) & =\frac{1}{2} a b \cdot \sin \alpha+\frac{1}{2} b c \cdot \sin \beta \\
\sin (\alpha+\beta) & =\frac{b}{c} \cdot \sin \alpha+\frac{b}{a} \cdot \sin \beta \\
\sin (\alpha+\beta) & =\sin \alpha \cos \beta+\cos \alpha \sin \beta
\end{aligned}
$$

## Practice 9

$\left(\sin 15^{\circ}\right)$ Show that $\sin 15^{\circ}=\frac{\sqrt{6}-\sqrt{2}}{4}$. Can you solve this problem using more than one method?
(1) Tip: This problem can be solved using the angle bisector theorem, just as we did to compute $\sin 18^{\circ}$ earlier. Meanwhile, it can also be solved using the Sum of Sine formula we just derived.
(i) Tip: We will show the $2^{\text {nd }}$ approach. It involves some techniques to simplify computation. You are encouraged to use the angle bisector theorem to solve this problem yourself.

Setting $\alpha=\beta$ in the sum of sine formula leads to the double angle formula:

$$
\sin 2 \alpha=2 \sin \alpha \cos \alpha
$$

Further setting $\alpha=15^{\circ}$ yields:

$$
\sin 30^{\circ}=2 \cdot \sin 15^{\circ} \cos 15^{\circ}
$$

Let $x=\sin 15^{\circ}$, we have

$$
\begin{aligned}
\frac{1}{2} & =2 x \sqrt{1-x^{2}} \\
1 & =4 x \sqrt{1-x^{2}} \\
1 & =16 x^{2}\left(1-x^{2}\right) \\
16 x^{4}-16 x^{2}+1 & =0 \\
\left(4 x^{2}-2\right)^{2} & =3 \\
x^{2} & =\frac{2 \pm \sqrt{3}}{4}
\end{aligned}
$$

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Because $0<\sin 15^{\circ}<\sin 30^{\circ}=\frac{1}{2} \Longrightarrow x^{2}<\frac{1}{4}$, it must hold that

$$
x^{2}=\frac{2-\sqrt{3}}{4} \Longrightarrow x=\frac{\sqrt{2-\sqrt{3}}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}
$$

(i) Tip: The technique used to simplify nested radical expression in the last step is discussed in the Power Calculation lecture and practice.

## Practice 10

(Trigonometry Form of Civa's Theorem)
As shown on the right, show that:

$$
\frac{\sin B A D}{\sin D A C} \cdot \frac{\sin A C F}{\sin F C B} \cdot \frac{\sin C B E}{\sin E B A}=1
$$


(i) Tip: When cevians and ratios are involved, the area method is always a good candidate to consider.

We have the following relationship:

$$
\frac{S_{\triangle A B O}}{S_{\triangle A O C}}=\frac{\frac{1}{2} \cdot A B \cdot A O \cdot \sin \triangle B A D}{\frac{1}{2} \cdot A O \cdot A C \cdot \sin \triangle D A C}=\frac{A B \cdot \sin \angle B A D}{A C \cdot \sin \angle D A C} \Longrightarrow \frac{\sin \angle B A D}{\sin \angle D A C}=\frac{S_{\triangle A B O} \cdot A C}{S_{\triangle O A C} \cdot A B}
$$

Similarly, we have:

$$
\begin{aligned}
& \frac{\sin \angle A C F}{\sin \angle F C B}=\frac{S_{\triangle A C O} \cdot B C}{S_{\triangle O C B} \cdot A C} \\
& \frac{\sin \angle C B E}{\sin \angle E B A}=\frac{S_{\triangle C B O} \cdot A B}{S_{\triangle O B A} \cdot B C}
\end{aligned}
$$

Multiplying these three equations gives us:

$$
\frac{\sin B A D}{\sin D A C} \cdot \frac{\sin A C F}{\sin F C B} \cdot \frac{\sin C B E}{\sin E B A}=\frac{S_{\triangle A B O} \cdot A C}{S_{\triangle O A C} \cdot A B} \cdot \frac{S_{\triangle A C O} \cdot B C}{S_{\triangle O C B} \cdot A C} \cdot \frac{S_{\triangle C B O} \cdot A B}{S_{\triangle O B A} \cdot B C}=1
$$

