Art of Thinking

The Coloring Method

Math for Gifted Students

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Art of Thinking

The Coloring Method

Instructions

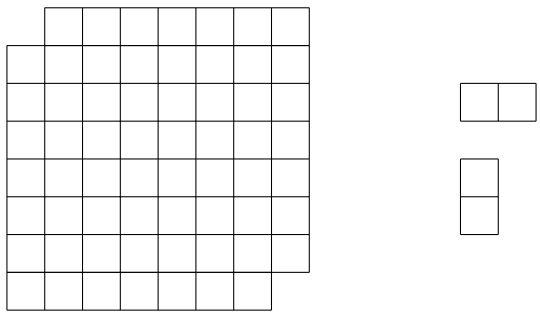
- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.* C_{100}^{50} *is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good one to practise?*
- *Complete the My Record section below before submission.*

My Comments and Notes

The Coloring Method

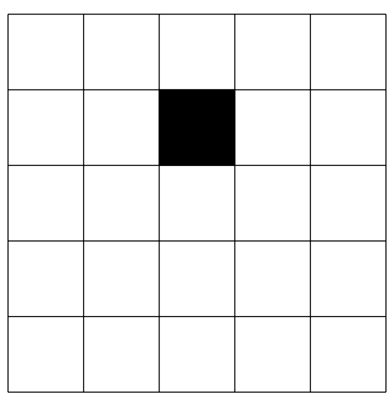
Practice 1

Joe cuts the top left corner and the bottom right corner of an 8×8 board, and then try to cover the remaining board using thirty-one 1×2 smaller pieces. Is it possible? Note: a smaller piece can be rotated, but cannot be further broken up.



Practice 2

There are 25 rooms in a house as shown. Every pair of neighboring rooms which shares a wall has a door connecting to each other. Is it possible to design a route to visit every room exactly once without entering the blacked out one? The route can start from any room.



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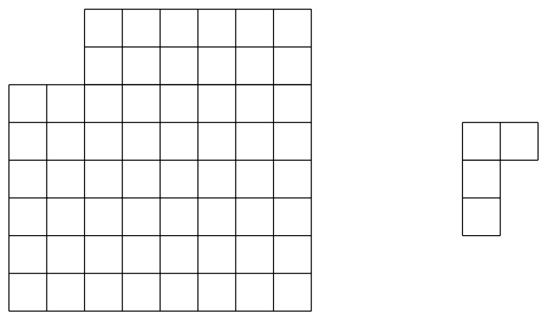
Practice 3

This time, Joe cuts a 2×2 corner off an 8×8 board, and then try to cover the remaining part using 15 smaller L-shaped grids made of 4 grids. Is it possible?



(Ref Ref 2745)

Practice 4

Show that it is impossible to cover an 8×8 square using fifteen 4×1 rectangles and one 2×2 square.

(Ref Ref 2759)

Practice 5

Show that among any 6 people in the world, there must exist 3 people who either know each other or do not know each other.

(Ref 2747: Hungarian Olympiad)

Practice 6

There are 6 points in the 3-D space. No three points are on the same line and no four points are one the same plane. Hence totally 15 segments can be created among these points. Show that if each of these 15 segments is colored either black or white, there must exist a triangle whose sides are of same color.

(Ref 2758: 1953 Putnam)

The Coloring Method

Practice 7

Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

(Ref 2749: 1964 IMO)

Practice 8

Is it possible to make these 1986×2 numbers: $1, 1, 2, 2, 3, 3, \cdots, 1986, 1986$ form a queue such that there is 1 number between two 1's, 2 numbers between two 2's, $\cdots$, 1986 numbers between two 1986's?

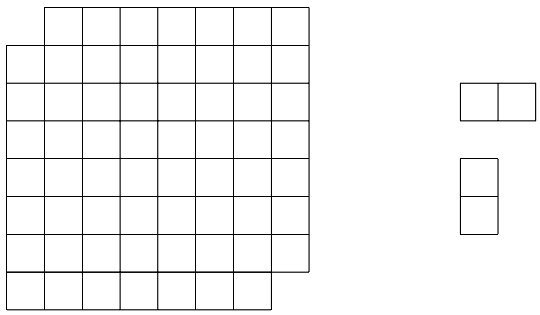
(Ref 2760: 1986 China Olympiad Winter Camp)

Reference Solutions

The Coloring Method

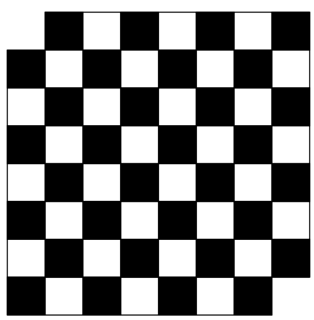
Practice 1

Joe cuts the top left corner and the bottom right corner of an 8×8 board, and then try to cover the remaining board using thirty-one 1×2 smaller pieces. Is it possible? Note: a smaller piece can be rotated, but cannot be further broken up.



It is impossible.

Let's color the remaining board in a way similar to a chess board.

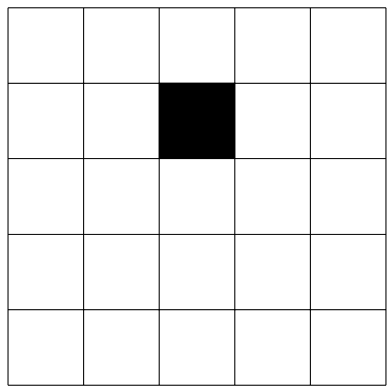


It is clear that there are totally 32 black grids and 30 white grids. Every 1×2 piece covers exactly 1 black and 1 white grid. It follows that 31 such pieces will cover exactly 31 black and 31 white grids. Hence we conclude this task is impossible.

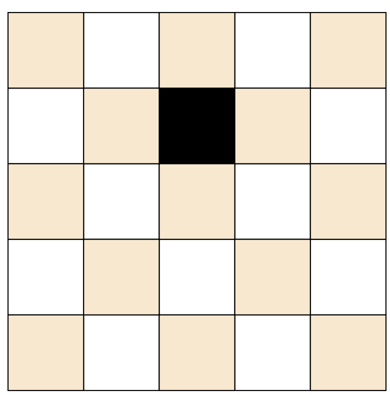
The Coloring Method

Practice 2

There are 25 rooms in a house as shown. Every pair of neighboring rooms which shares a wall has a door connecting to each other. Is it possible to design a route to visit every room exactly once without entering the blacked out one? The route can start from any room.



Again, it is impossible. We still use a similar coloring scheme here.



There are 13 shaded rooms and 11 white rooms, respectively. The difference is 2. However, a route will always go through a shaded room and a white room in turn. Therefore the difference in number of each type rooms visited can only be 0 or 1, but never 2. This means no such route is possible to visit 13 shaded rooms and 11 white rooms without repeating.

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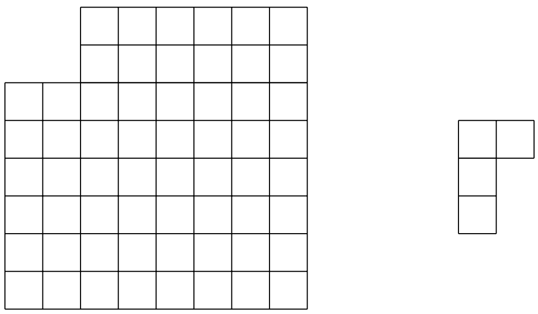
The Coloring Method

Practice 3

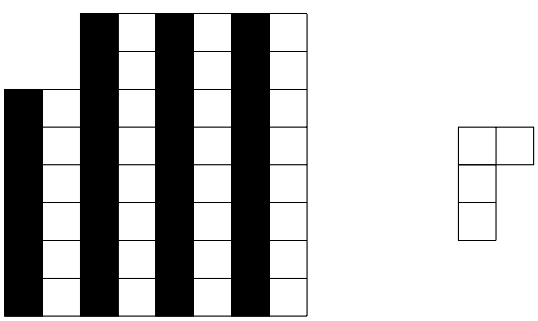
This time, Joe cuts a 2×2 corner off an 8×8 board, and then try to cover the remaining part using 15 smaller L-shaped grids made of 4 grids. Is it possible?



(Ref Ref 2745)

It is also impossible.

This time, instead of painting adjacent grids, we paint adjacent columns using alternating colors.



Then we note that, regardless of how to place the $L-$shape piece, it will always cover either one or three black grids. Therefore 15 such pieces must cover an odd number of black grids in total. However the number of black grids is an even number on this board. Therefore this is another impossible mission.

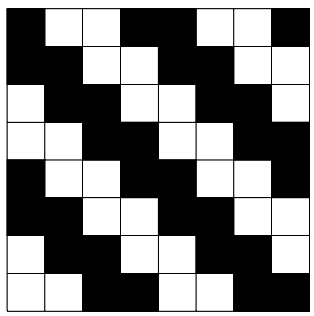
Tip: An appropriate coloring scheme plays a key role in solving such type of problems.

Practice 4

Show that it is impossible to cover an 8×8 square using fifteen 4×1 rectangles and one 2×2 square.

(Ref Ref 2759)

In order to solve this problem, we turn to the following coloring scheme.



Now we note that an 1×4 piece will always cover two white and two black grids. This implies that 15 such pieces will cover 30 black and 30 white ones. Remaining grids must be 2 black and 2 white. However a 2×2 piece can only cover either 3 black + 1 white, or 1 black + 3 white.

Therefore we conclude it is an impossible task to cover the whole board using fifteen 4×4 pieces and one 2×2 piece.

Tip: Coloring scheme rocks.

Practice 5

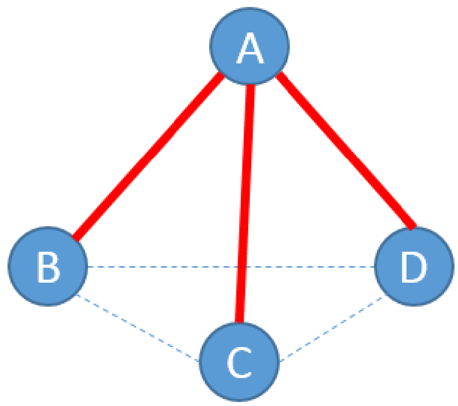
Show that among any 6 people in the world, there must exist 3 people who either know each other or do not know each other.

(Ref 2747: Hungarian Olympiad)

Tip: The coloring method is often used together with the Pigeonhole principle.

Let's 6 nodes to represent any six people, and name them A, B, C, D, E, and F. If two people know each other, we connect these two nodes using a green line. Otherwise, if these two people do not know each other, we connect these two nodes using a red line. Then the original problem is equivalent to show that there must exist a triangle whose three sides are of the same color.

Now let's consider node A. Because it has 5 connections with the other nodes, at least three connections are of the same color. Without loss of generality, let's assume AB, AC and AD are all red.



Next, let's consider the connections among B, C, and D. If any of them is red, we find a red triangle. Otherwise, if none of them is red which means all of them are green, then we find $\triangle BCD$ is a green triangle. Either way, there must exist a triangle whose three sides are of same color.

Tip: This problem relates to concept of complete graph in graph theory.

Practice 6

There are 6 points in the 3-D space. No three points are on the same line and no four points are one the same plane. Hence totally 15 segments can be created among these points. Show that if each of these 15 segments is colored either black or white, there must exist a triangle whose sides are of same color.

(Ref 2758: 1953 Putnam)

This is exactly the same type of question as the previous one, isn't it?

Tip: Quite often, problems may look different, but are essentially the same.

Practice 7

Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

(Ref 2749: 1964 IMO)

Tip: This problem is essentially the same as the previous two problems.

Let's use seventeen nodes to represent these seventeen people. We also use three different colors, black, red and green, to represent these three topics. Thus the original problem is equivalent to show that there must exist a triangle whose three sides are of the same color.

Let's consider node N_1. Because it needs to connect to other sixteen node using three different colors, at least 6 of them must share the same color. Without loss of generality, let's assume the connection between N_1 and $N_2, N_3, N_4, N_5, N_6, N_7$ are black. It follows that if any connections between these six nodes is black, we find a black triangle.

Otherwise if none of them is black, we find the original problems becomes to show that there must exist either a green triangle or a red triangle, if all the connections among these six nodes are either green or red. This is exactly the same as the previous *six people* problem.

Hence, we know the original claim holds.

Tip: Do you see the similarity among these problems?

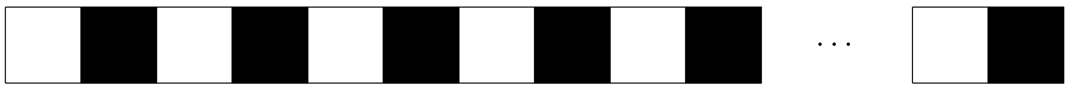
Practice 8

Is it possible to make these 1986×2 numbers: $1, 1, 2, 2, 3, 3, \cdots, 1986, 1986$ form a queue such that there is 1 number between two 1's, 2 numbers between two 2's, $\cdots$, 1986 numbers between two 1986's?

(Ref 2760: 1986 China Olympiad Winter Camp)

The answer is impossible.

Let's color these 2×1986 positions white and black in turn, as shown below.



It is clear that a pair of even numbers must occupy one white and one black positions because the number of intervals between them is even. On the other hand, a pair of odd numbers will occupy two positions of the same color because there must be odd number of positions between them.

Let's assume:

1. the 993 pairs of even numbers occupy $W_e = 993$ white positions and $B_e = 993$ black positions.
2. the 993 pairs of odd numbers occupy $W_o = 2a$ white positions and $B_o = 2b$ black positions, where a and b are two non-negative integers such that $a + b = 993$. Because 993 is odd, we find $a \neq b$.

It follows that total number of white positions is $W_e + W_o = 993 + 2a$, and the total number of black positions is $B_e + B_o = B_e + 2b$. Because $a \neq b$, it will lead to the number of white positions does not equal the number of black positions.

However, by the construction of our coloring scheme, there are same numbers of white and black positions. Hence, we conclude that it is impossible.