

# Vieta's Formula

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# Vieta's Theorem

- ➔ Describes the relation between a polynomial's roots and its coefficients
- ➔ A must-master technique
- ➔ The key is **NOT** to solve the equation directly



# Quadratic Vieta's Formula

Let  $x_1$  and  $x_2$  be the two roots of quadratic equation  $ax^2 + bx + c = 0$ , then

$$x_1 + x_2 = -\frac{b}{a} \qquad x_1 \cdot x_2 = \frac{c}{a}$$

A simple but *silly* proof

$$\left\{ \begin{array}{l} x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \end{array} \right. \longrightarrow \left\{ \begin{array}{l} x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a} \\ x_1 \cdot x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{c}{a} \end{array} \right.$$

Why this is a *silly* proof?

# Another Proof

example

Find a quadratic equation whose roots are 1 and 2.

Solution:  $(x - 1)(x - 2) = 0 \implies k(x - 1)(x - 2) = 0$ , where  $k \neq 0$ .

$$\begin{array}{ccc} \Downarrow & & \\ x^2 - 3x + 2 = 0 & \implies & 1 + 2 = 3 \quad \checkmark \quad 1 \times 2 = 2 \quad \checkmark \end{array}$$

Vieta's Formula

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $ax^2 + bx + c = 0$ , then  $x_1 + x_2 = -\frac{b}{a}$ ,  $x_1x_2 = \frac{c}{a}$ .

$$ax^2 + bx + c = 0 \iff a(x - x_1)(x - x_2) = 0 \iff ax^2 - a(x_1 + x_2)x + ax_1x_2 = 0$$

$$\left\{ \begin{array}{l} b = -a(x_1 + x_2) \\ c = ax_1x_2 \end{array} \right. \implies \left\{ \begin{array}{l} x_1 + x_2 = -\frac{b}{a} \\ x_1 \cdot x_2 = \frac{c}{a} \end{array} \right.$$

Why is this proof better?

# Example

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

① Find the value of  $x_1^2 + x_2^2$ .

Expression containing roots

② Find a quadratic equation whose roots are  $x_1^2$  and  $x_2^2$ .

Equation construction

③ Find the value of  $\frac{1}{x_1+1} + \frac{1}{x_2+1}$ .

More expression

④ Write a recurrence relation for sequence  $y_n = x_1^n + x_2^n$ .

Advanced topic

⑤ Find the value of  $x_1^3 + x_2^3$ .

More expression

## The key is **NOT** to solve the roots!

The simple equation given in this example is for illustration purpose so you can easily check your result.



# Example Solution

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

① Find the value of  $x_1^2 + x_2^2$

Expression containing roots

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (3)^2 - 2 \times (2) = 5$$

Convert the target expression to  $(x_1 + x_2)$  and  $(x_1 \cdot x_2)$  using polynomial transformation.



# Example Solution

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

② Find a quadratic equation whose roots are  $x_1^2$  and  $x_2^2$ .

Equation construction



It is equivalent to finding the value of  $x_1^2 + x_2^2$  and  $x_1^2 x_2^2$ . Why?

$$\begin{cases} x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (3)^2 - 2 \times (2) = 5 \\ x_1^2 x_2^2 = (x_1 x_2)^2 = (2)^2 = 4 \end{cases}$$

$\therefore$  one desired equation is  $x^2 - 5x + 4 = 0$ .

Can you verify the result?

# Example Solution

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

③ Find the value of  $\frac{1}{x_1+1} + \frac{1}{x_2+1}$ .

More expression

Solution 1: 
$$\frac{1}{x_1 + 1} + \frac{1}{x_2 + 1} = \frac{(x_2 + 1) + (x_1 + 1)}{(x_1 + 1)(x_2 + 1)} = \frac{(x_1 + x_2) + 2}{x_1x_2 + (x_1 + x_2) + 1} = \frac{3 + 2}{2 + 3 + 1} = \frac{5}{6}$$

Solution 2:

$x_1$  and  $x_2$  are the roots of

$$x^2 - 3x + 2 = 0$$

➡  $(x_1+1)$  and  $(x_2 + 1)$  are the roots of

$$(x - 1)^2 - 3(x - 1) + 2 = 0 \text{ or } x^2 - 5x + 6 = 0$$

➡  $\frac{1}{x_1+1}$  and  $\frac{1}{x_2+1}$  are the roots of

$$\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) + 6 = 0 \text{ or } 6x^2 - 5x + 1 = 0$$

$$\therefore \frac{1}{x_1 + 1} + \frac{1}{x_2 + 1} = -\left(\frac{-5}{6}\right) = \frac{5}{6} \quad \checkmark$$

# Example Solution

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

④ Write a recurrence relation for sequence  $y_n = x_1^n + x_2^n$ .

Advanced topic

The answer is:  $y_{n+2} - 3y_{n+1} + 2y_n = 0$ , and  $y_0 = 2$  and  $y_1 = 3$ .

This is because

$$x_1^2 - 3x_1 + 2 = 0 \xrightarrow{\text{multiply } x_1^n} x_1^{n+2} - 3x_1^{n+1} + 2x_1^n = 0$$

$$x_2^2 - 3x_2 + 2 = 0 \xrightarrow{\text{multiply } x_2^n} x_2^{n+2} - 3x_2^{n+1} + 2x_2^n = 0$$

$$\rightarrow (x_1^{n+2} + x_2^{n+2}) - 3(x_1^{n+1} + x_2^{n+1}) + 2(x_1^n + x_2^n) = 0$$

$$\rightarrow y_{n+2} - 3y_{n+1} + 2y_n = 0$$

# Example

Let  $x_1$  and  $x_2$  be two roots of quadratic equation  $x^2 - 3x + 2 = 0$ .

⑤ Find the value of  $x_1^3 + x_2^3$ .

More expression

Solution 1:

$$x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = (3)^3 - 3 \times (2) \times (3) = 9$$

Solution 2:

Let  $y_n = x_1^n + x_2^n$ , then  $y_{n+2} - 3y_{n+1} + 2y_n = 0$ , or  $y_{n+2} = 3y_{n+1} - 2y_n$ .

$$y_0 = x_1^0 + x_2^0 = 2$$

$$y_1 = x_1^1 + x_2^1 = 3 \text{ by Vieta's theorem}$$

$$\Rightarrow y_2 = 3y_1 - 2y_0 = 3 \times 3 - 2 \times 2 = 5$$



$$\Rightarrow y_3 = 3y_2 - 2y_1 = 3 \times 5 - 2 \times 3 = 9$$



# $n^{\text{th}}$ Degree Equation Vieta's Formula

Let  $x_1, \dots, x_n$  be the roots of equation  $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$ , then

$$\left\{ \begin{array}{l} \sum_{i=1}^n x_i = x_1 + x_2 + x_3 + \dots + x_n = -a_{n-1} \\ \sum_{i \neq j} x_i x_j = x_1 x_2 + \dots + x_1 x_n + x_2 x_3 + \dots = a_{n-2} \\ \dots \\ x_1 x_2 x_3 \dots x_{n-1} x_n = (-1)^n a_0 \end{array} \right.$$

	expression	number of terms	sign
$\left\{ \begin{array}{l} \sum_{i=1}^n x_i = x_1 + x_2 + x_3 + \dots + x_n = -a_{n-1} \\ \sum_{i \neq j} x_i x_j = x_1 x_2 + \dots + x_1 x_n + x_2 x_3 + \dots = a_{n-2} \\ \dots \\ x_1 x_2 x_3 \dots x_{n-1} x_n = (-1)^n a_0 \end{array} \right.$	sum of products of 1 root	$C_n^1 = n$	
	sum of products of 2 roots	$C_n^2$	
	sum of products of $n$ roots	$C_n^n = 1$	

What if the coefficient of the first term is not 1?