Vieta's Formula

Vieta's Theorem

- Describes the relation between a polynomial's roots and its coefficients
- A must-master technique
- The key is NOT to solve the equation directly

Quadratic Vieta's Formula

Let x_1 and x_2 be the two roots of quadratic equation $ax^2 + bx + c = 0$, then

$$x_1 + x_2 = -\frac{b}{a} \qquad x_1 \cdot x_2 = \frac{c}{a}$$

A simple but *silly* proof

$$\begin{cases} x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \\ x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{a} \\ x_1 \cdot x_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{c}{a} \end{cases}$$

Why this is a *silly* proof?

Another Proof

example

Find a quadratic equation whose roots are 1 and 2.

Solution: $(x-1)(x-2)=0 \implies k(x-1)(x-2)=0$, where $k \neq 0$.

$\Downarrow$

$x^2-3x+2=0 \implies 1+2=3$ ✓ $1\times 2=2$ ✓

Vieta's Formula

Let x_1 and x_2 be two roots of quadratic equation $ax^2+bx+c=0$, then $x_1+x_2=-\frac{b}{a}$, $x_1x_2=\frac{c}{a}$.

$$ax^2+bx+c=0 \iff a(x-x_1)(x-x_2)=0 \iff ax^2-a(x_1+x_2)x+ax_1x_2=0$$

$$\begin{cases} b=-a(x_1+x_2) \\ c=ax_1x_2 \end{cases} \implies \begin{cases} x_1+x_2=-\dfrac{b}{a} \\ x_1\cdot x_2=\dfrac{c}{a} \end{cases}$$

Why is this proof better?

Example

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

1. Find the value of $x_1^2 + x_2^2$. — Expression containing roots
2. Find a quadratic equation whose roots are x_1^2 and x_2^2 . — Equation construction
3. Find the value of $\frac{1}{x_1+1} + \frac{1}{x_2+1}$. — More expression
4. Write a recurrence relation for sequence $y_n = x_1^n + x_2^n$. — Advanced topic
5. Find the value of $x_1^3 + x_2^3$. — More expression

The key is NOT to solve the roots!

The simple equation given in this example is for illustration purpose so you can easily check your result.

Example Solution

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

① Find the value of $x_1^2 + x_2^2$

Expression containing roots

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (3)^2 - 2 \times (2) = 5$$

Convert the target expression to $(x_1 + x_2)$ and $(x_1 \cdot x_2)$ using polynomial transformation.

Example Solution

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

(2) Find a quadratic equation whose roots are x_1^2 and x_2^2 .

Equation construction

It is equivalent to finding the value of $x_1^2 + x_2^2$ and $x_1^2 x_2^2$. Why?

$$\begin{cases} x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (3)^2 - 2 \times (2) = 5 \\ x_1^2 x_2^2 = (x_1x_2)^2 = (2)^2 = 4 \end{cases}$$

$\therefore$ one desired equation is $x^2 - 5x + 4 = 0$.

Can you verify the result?

Example Solution

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

③ Find the value of $\frac{1}{x_1+1} + \frac{1}{x_2+1}$.

More expression

Solution 1: $$\frac{1}{x_1+1} + \frac{1}{x_2+1} = \frac{(x_2+1)+(x_1+1)}{(x_1+1)(x_2+1)} = \frac{(x_1+x_2)+2}{x_1x_2+(x_1+x_2)+1} = \frac{3+2}{2+3+1} = \frac{5}{6}$$

Solution 2:

x_1 and x_2 are the roots of $x^2 - 3x + 2 = 0$

➡ (x_1+1) and (x_2+1) are the roots of $(x-1)^2 - 3(x-1) + 2 = 0$ or $x^2 - 5x + 6 = 0$

➡ $\frac{1}{x_1+1}$ and $\frac{1}{x_2+1}$ are the roots of $\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) + 6 = 0$ or $6x^2 - 5x + 1 = 0$

$$\therefore \frac{1}{x_1+1} + \frac{1}{x_2+1} = -\left(\frac{-5}{6}\right) = \frac{5}{6}$$ ✔

Example Solution

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

(4) Write a recurrence relation for sequence $y_n = x_1^n + x_2^n$.

Advanced topic

The answer is: $y_{n+2} - 3y_{n+1} + 2y_n = 0$, and $y_0 = 2$ and $y_1 = 3$.

This is because

$$x_1^2 - 3x_1 + 2 = 0 \xRightarrow[multiply\ \ x_1^n]{} x_1^{n+2} - 3x_1^{n+1} + 2x_1^n = 0$$

$$x_2^2 - 3x_2 + 2 = 0 \xRightarrow[multiply\ \ x_2^n]{} x_2^{n+2} - 3x_2^{n+1} + 2x_2^n = 0$$

➡ $(x_1^{n+2} + x_2^{n+2}) - 3(x_1^{n+1} + x_2^{n+1}) + 2(x_1^n + x_2^n) = 0$

➡ $y_{n+2} - 3y_{n+1} + 2y_n = 0$

Example

Let x_1 and x_2 be two roots of quadratic equation $x^2 - 3x + 2 = 0$.

(5) Find the value of $x_1^3 + x_2^3$.

More expression

Solution 1:

$$x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = (3)^3 - 3 \times (2) \times (3) = 9$$

Solution 2:

Let $y_n = x_1^n + x_2^n$, then $y_{n+2} - 3y_{n+1} + 2y_n = 0$, or $y_{n+2} = 3y_{n+1} - 2y_n$.

$y_0 = x_1^0 + x_2^0 = 2$

$y_1 = x_1^1 + x_2^1 = 3$ by Vieta's theorem

$\Rightarrow y_2 = 3y_1 - 2y_0 = 3 \times 3 - 2 \times 2 = 5$

$\Rightarrow y_3 = 3y_2 - 2y_1 = 3 \times 5 - 2 \times 3 = 9$

n^{th} Degree Equation Vieta's Formula

Let $x_1, \cdots, x_n$ be the roots of equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, then

$$\begin{cases} \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n = -a_{n-1} \\ \sum_{i \neq j} x_i x_j = x_1 x_2 + \cdots + x_1 x_n + x_2 x_3 + \cdots = a_{n-2} \\ \cdots \\ x_1 x_2 x_3 \cdots x_{n-1} x_n = (-1)^n a_0 \end{cases}$$

	expression	number of terms	sign
$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n = -a_{n-1}$	sum of products of 1 root	$C_n^1 = n$	−
$\sum_{i \neq j} x_i x_j = x_1 x_2 + \cdots + x_1 x_n + x_2 x_3 + \cdots = a_{n-2}$	sum of products of 2 roots	C_n^2	+
$\cdots$			
$x_1 x_2 x_3 \cdots x_{n-1} x_n = (-1)^n a_0$	sum of products of n roots	$C_n^n = 1$	

What if the coefficient of the first term is not 1?