Indeterminate Equation

Assessment

Math for Gifted Students

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Assessment

Indeterminate Equation

Instructions

- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.* C_{100}^{50} *is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good one to practise?*
- *Complete the My Record section below before submission.*

My Comments and Notes

Indeterminate Equation

Practice 1

Find all ordered integer pairs (x, y) such that $x + xy + y = 8$.

Practice 2

Solve in integers the equation $41x + 37y = 13$.

Practice 3

Solve in positive integers the following equations:

(i) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$

(ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

(iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

Practice 4

Solve the equation $x^2 + y^2 = 6x - 4y - 13$.

Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?

Practice 6

Find all the right triangles that satisfy the following two conditions:

(i) the lengths of all its three sides are integers, and

(ii) its area and perimeter are numerically equal

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Practice 7

Solve in positive integers the equation $y^2 = x^2 + x + 1$.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Practice 9

Solve in integers the equation: $x^2 + y^2 = 2015$.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

Practice 11

Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

Reference Solutions

Indeterminate Equation

Practice 1

Find all ordered integer pairs (x, y) such that $x + xy + y = 8$.

Tip: The factorization method and integer divisibility.

Solution 1: Polynomial Factorization

The given equation can be re-written as:

$$(x+1)(y+1) = 9$$

Because both x and y are integers, both $(x+1)$ and $(y+1)$ are integers too. It follows that they must be paired divisors of 9:

$x+1$	$y+1$	x	y
1	9	0	8
3	3	2	2
9	1	8	0
-1	-9	-2	-10
-3	-3	-4	-4
-9	-1	-10	-2

Therefore we conclude there are totally 6 solutions.

Solution 2: Integer Divisibility

Rearrange the given equation as an equation with respect to x: $(y+1)x = 8 - y$. Hence:

$$x = \frac{8-y}{y+1} = \frac{9-(1+y)}{y+1} = \frac{9}{y+1} - 1 \tag{1}$$

Because x is an integer, $\dfrac{9}{y+1}$ must be an integer. This follows that $(y+1)$ must be a divisor of 9, or

$$\begin{aligned} y+1 &= -9, -3, -1, 1, 3, 9 \\ y &= -10, -4, -2, 0, 2, 8 \end{aligned}$$

Setting these values to *Equation 1*, respectively, leads to

$$x = -2, -4, -10, 8, 2, 0$$

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Practice 2

Solve in integers the equation $41x + 37y = 13$.

Tip: The Euclidean method, and the $ax + by = 1$ and $ax + by = c$ patterns.

First, let's solve

$$41x + 37y = 1 \tag{2}$$

This can be done by the Euclidean method. We have

$$\begin{aligned} 41 &= 37 \times 1 + 4 \\ 37 &= 4 \times 9 + 1 \end{aligned}$$

Therefore

$$1 = 37 - 4 \times 9 = 37 - (41 - 37 \times 1) \times 9 = -41 \times 9 + 37 \times 10$$

This means *Equation 2* has one solution $(-9, 10)$, and its general solution is given by:

$$\begin{cases} x &= -9 + 37t \\ y &= 10 - 41t \end{cases} \tag{3}$$

where t is an integer parameter.

It follows that

$$\begin{cases} x &= -9 \times 13 + 37 \times 13t &= -117 + 481t \\ y &= 10 \times 13 - 41 \times 13t &= 130 - 533t \end{cases} \tag{4}$$

is the solution to the original question $41x + 37y = 13$.

Practice 3

Solve in positive integers the following equations:

(i) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3}$

(ii) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$

(iii) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{3}{5}$

Indeterminate Equation

(i)

Tip: The factorization method, and the $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$ *pattern.*

The given equation can be rewritten as $(x-3)(y-3)=9$. Therefore both $(x-3)$ and $(y-3)$ must be divisors of 16. Without loss of generality, let's assume $x-3 \geq y-3$. This follows that one of the following must hold:

$$\begin{cases} x-3 &= 9 \\ y-3 &= 1 \end{cases} \quad \text{or} \quad \begin{cases} x-3 &= 3 \\ y-3 &= 3 \end{cases}$$

Solving these two systems lead to $(x,y)=(12,4),(6,6)$. Hence all the solutions are

$$(x,y)=(12,4),(6,6),(4,12)$$

(ii)

Tip: The inequality method, and the $\frac{1}{x}+\frac{1}{y}=\frac{m}{n}$ *pattern.*

By symmetry, let's assume $x \leq y$. Hence

$$\frac{1}{x} \geq \frac{1}{y}$$

It follows that,

$$\frac{1}{x} \geq \frac{1}{2} \times \frac{5}{6} = \frac{5}{12} \tag{5}$$

or $x \leq 2$.

Testing $x=1,2$ respectively finds $(2,3)$ is one solution. Therefore all the solutions to the given equations are

$$(x,y)=(2,3),(3,2)$$

Quiz: Can you use this method to solve (i) above?

(iii)

Tip: The inequality method, and the $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{m}{n}$ *pattern.*

By the symmetrical argument, let's assume $0<x \leq y \leq z$. It follows:

$$\frac{1}{x} < \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \frac{3}{x}$$

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Then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{5} \implies \frac{1}{x}<\frac{3}{5}\leq\frac{3}{x} \implies 2\leq x\leq 5$.

Now we proceed with casework:

If $x=2$, then $\frac{1}{y}+\frac{1}{z}=\frac{3}{5}-\frac{1}{2}=\frac{1}{10}$.

If $x=3$, then $\frac{1}{y}+\frac{1}{z}=\frac{3}{5}-\frac{1}{3}=\frac{4}{15}$.

If $x=4$, then $\frac{1}{y}+\frac{1}{z}=\frac{3}{5}-\frac{1}{4}=\frac{7}{20}$.

If $x=5$, then $\frac{1}{y}+\frac{1}{z}=\frac{3}{5}-\frac{1}{5}=\frac{2}{5}$.

The equivalent equation in every case is in the form of:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n} \quad \text{or} \quad \frac{1}{x}+\frac{1}{y}=\frac{m}{n}$$

They can all be solved by using the techniques that are presented in (i) and (ii) above. Solving these equations leads to the following solutions under the assumption $0<x\leq y\leq z$.

(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Therefore, all the solutions are just distinct permutations of the above set.

Practice 4

Solve the equation $x^2+y^2=6x-4y-13$.

Tip: The sum of square method.

The given equation is equivalent to:

$$(x-3)^2+(y+2)^2=0$$

Because squares cannot be negative, the only possibility to make this equation hold is $(x,y)=(3,-2)$.

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Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?

Tip: Property of square numbers.

It is obvious that $(x^2 + 3)$ must be a multiple of 5. It follows that the unit digit of $(x^2 + 3)$ must be either 0 or 5. Equivalently, x^2 must end with 8 or 3. However no square number can end with 8 or 3. Hence this equation is not solvable.

Practice 6

Find all the right triangles that satisfy the following two conditions:

(i) the lengths of all its three sides are integers, and

(ii) its area and perimeter are numerically equal

Tip: The Pythagorean triplet formula.

By the Pythagorean triplet formula, the lengths of three sides can be written as:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}$$

where m and n are two positive integers.

If its area and perimeter equal in values, the following must hold:

$$\frac{1}{2}(m^2 - n^2)(2mn) = (m^2 - n^2) + 2mn + (m^2 + n^2)$$

It follows that:

$$\begin{aligned} (m^2 - n^2)(mn) &= 2m^2 + 2mn \\ (m+n)(m-n)mn &= 2m(m+n) \\ (m-n)n &= 2 \end{aligned}$$

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This is a basic indeterminate equation that can be solved using the factorization method.

$$\begin{cases} m-n &= 1 \\ n &= 2 \end{cases} \quad \text{or} \quad \begin{cases} m-n &= 2 \\ n &= 1 \end{cases}$$

We find $(m, n) = (3, 2)$ or $(3, 1)$.

Setting these values into the Pythagorean triplet formula produces two such triangles: 5-12-13 and 6-8-10.

Practice 7

Solve in positive integers the equation $y^2 = x^2 + x + 1$.

Tip: The squeeze method.

From the given equation and the condition $x > 0$, it is easy to see that

$$y^2 < x^2 < (x+1)^2$$

Note that x and $(x+1)$ are two consecutive integers. Therefore it is impossible to have another integer whose square is between x^2 and $(x+1)^2$.

Hence, we conclude that no solution is possible.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Tip: The quadratic method.

Let $u = \frac{x}{y}$, then the given problem is equivalent to:

$$u + \frac{15}{4u} = k$$

where k is an integer. It is obvious that u is a positive rational number because both x and y are positive integers.

Rewriting this relationship leads to:

$$4u^2 - 4ku + 15 = 0 \tag{6}$$

Because *Equation 6* is solvable in rational number, its discriminant must be a square number. Let

$$\Delta = 16k^2 - 4 \times 4 \times 15 = n^2$$

where n is an integer. Or:

$$16k^2 - n^2 = 240$$

Clearly, n^2 is a multiple of $16 = 4^2$, setting $n = 4m$ leads to:

$$\begin{aligned} 16k^2 - 16m^2 &= 240 \\ k^2 - m^2 &= 15 \\ (k+m)(k-m) &= 15 \end{aligned} \tag{7}$$

Equation 7 can be solved by the factorization method. Because both k and m are positive integers, we have $k + m > k - m$. Consequently, one of the two systems must hold:

$$\begin{cases} k+m &= 15 \\ k-m &= 1 \end{cases} \quad \text{or} \quad \begin{cases} k+m &= 5 \\ k-m &= 3 \end{cases}$$

Solving the above two systems leads to

$$(k, m) = (8, 7), (4, 1)$$

Setting $k = 8$ to the quadratic formula of *Equation 6*:

$$u = \frac{4 \times 8 \pm \sqrt{(4 \times 8)^2 - 4 \times 4 \times 15}}{2 \times 4} = 4 \pm \frac{7}{2} = \frac{15}{2}, \frac{1}{2}$$

Setting $k = 4$ leads:

$$u = \frac{4 \times 4 \pm \sqrt{(4 \times 4)^2 - 4 \times 4 \times 15}}{2 \times 4} = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}$$

Therefore, we conclude there are four solutions:

$$(x, y) = (15, 2), (1, 2), (5, 2) \text{ and } (3, 2)$$

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Practice 9

Solve in integers the equation: $x^2 + y^2 = 2015$.

Tip: Number theory / Frequently used MOD conclusions.

Taking MOD 4 on both sides leads to

$$x^2 + y^2 = 2015 \equiv 3 \pmod 4$$

However this relationship cannot hold. Therefore the original equation is not solvable in integers.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

Tip: The MOD method

First, let's show x, y, and z must be all even.

Taking (mod 4) on both sides of the equation leads to:

$$(-1)^x + 0 \equiv 1^z \pmod 4$$

Clearly, this relationship can only hold if x is even.

Next, taking (mod 3) on both sides yields:

$$0 + 1^y \equiv (-1)^z \pmod 3$$

Therefore, z must be even too.

As such, let $x = 2k$, $z = 2p$, and note $4^y = (2^y)^2$, the original equation becomes:

$$(3^k)^2 + (2^y)^2 = (5^p)^2$$

Therefore $(3^k, 2^y, 5^p)$ forms a Pythagorean triplet. Hence, there exist positive integers m and n such that [1]:

$$\begin{cases} 3^k &= m^2 - n^2 \\ 2^y &= 2mn \\ 5^p &= m^2 + n^2 \end{cases}$$

[1] Note 3^k is an odd number. Therefore it cannot equal $2mn$

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Because $2^y = 2mn$, both m and n must be some power of 2. Let $m = 2^t$ and $n = 2^s$ where t and s are non-negative integers satisfying $t + s = y - 1$. Note $m > n \implies t > s$.

It follows that:

$$\begin{cases} 3^k &= m^2 - n^2 &= 2^{2t} - 2^{2s} &= 2^{2s}(2^{2(t-s)} - 1) \\ 5^p &= m^2 + n^2 &= 2^{2t} + 2^{2s} &= 2^{2s}(2^{2(t-s)} + 1) \end{cases}$$

Because neither 3^k nor 5^p is divisible by 2, we conclude 2^{2s} must equal 1. This means $s = 0$, and $2^{2(t-s)} = 4$ or $t = 1$. It is followed by $k = p = 1$.

Hence, the given equation has only one positive integer solution: $x = y = z = 2$.

Practice 11

Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Tip: The infinite descent method.

If there exists such a positive integer solution (x, y, z), then x must be even. Let $x = 2x_1$:

$$\begin{aligned} (2x_1)^3 + 2y^3 &= 4z^3 \\ 4x_1^3 + y^3 &= 2z^3 \end{aligned}$$

This means y must be even too. Let $y = 2y_1$:

$$\begin{aligned} 4x_1^3 + (2y_1)^3 &= 2z^3 \\ 2x_1^3 + 4y_1^3 &= z^3 \end{aligned}$$

This in turn shows z is also even. Let be $z = 2z_1$:

$$\begin{aligned} 2x_1^3 + 4y_1^3 &= (2z_1)^3 \\ x_1^3 + 2y_1^3 &= 4z_1^3 \end{aligned}$$

This last equation is in the same form of the original one. Hence, we conclude if (x, y, z) is a positive integer solution, x, y, and z must be all even, and $(x_1, y_1, z_1) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ will be a solution too. It is clear that the process of $(x, y, z) \implies (x_1, y_1, z_1)$ is repeatable. Therefore an infinitive decreasing solution series can be constructed, which is impossible by the principle of infinite descent. Thus, the equation is unsolvable in positive integers.

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Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

Tip: The Pell's equation.

Let three sides' lengths be $z-1$, z, and $z+1$, respectively. Then by the Heron's formula, the triangle's area is given by:

$$S = \sqrt{\frac{3}{2}z \times \left(\frac{3}{2}z - (z-1)\right)\left(\frac{3}{2}z - z\right)\left(\frac{3}{2}z - (z+1)\right)}$$

$$= \frac{z}{4}\sqrt{3(z^2-4)} \quad (8)$$

If S is an integer, then $3(z^2-4)$ must be a square number. Let

$$3(z^2-4) = 3w^2$$

In addition, from *Equation 8*, it is clear that z must be even because, otherwise, both z and $\sqrt{3(z^2-4)}$ will be odd. This will make S a non-integer.

Letting $z = 2x$ leads to $4x^2 - 4 = 3w^2$. This means that w must be even too. Letting $w = 2y$ and simplifying yield:

$$x^2 - 3y^2 = 1$$

This is a Pell's equation. Its fundamental solution is

$$(x, y) = (2, 1)$$

and general solution is given by:

$$\begin{cases} x_n = \dfrac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2} \\ y_n = \dfrac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} \end{cases} \quad (9)$$

As a result, there exist infinitely many such triangles. Three sides are $(2x_n - 1, 2x_n, , 2x_n + 1)$ where x_n is given by *Equation 9*, and area is $3 \cdot x_n \cdot y_n$.

The three smallest such triangles can be obtained by setting $n = 1, 2,$ and 3, respectively:

$$(3, 4, 5), (13, 14, 15), (51, 52, 53)$$

Answer Keys

Practice 1: $(x, y) = (0, 8), (8, 0), (2, 2), (-2, -10), (-10, -2), (-4, -4)$

Practice 2:

$$\begin{cases} x &= -9 \times 13 + 37 \times 13t &= -117 + 481t \\ y &= 10 \times 13 - 41 \times 13t &= 130 - 533t \end{cases}$$

where t is an integer parameter.

Practice 3:

(i) $(x, y) = (2, 3), (3, 2)$

(ii) $(x, y) = (12, 4), (6, 6), (4, 12)$

(iii) Permutation of the following sets: (2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Practice 4: $(x, y) = (3, -2)$

Practice 5: No solution exists.

Practice 6: 5-12-13 and 6-8-10.

Practice 7: No solution exits.

Practice 8: $(x, y) = (15, 2), (1, 2), (5, 2)$ and $(3, 2)$

Practice 9: No solution exits.

Practice 10: $(x, y, z) = (2, 2, 2)$

Practice 11: No solution exits.

Practice 12: There exist infinitely many such triangles. Refer to the reference solution.