## **Indeterminate Equation**

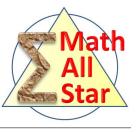
Assessment



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## Assessment

# Indeterminate Equation



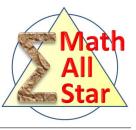
## Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g.  $C_{100}^{50}$  is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good one to practise?
- Complete the My Record section below before submission.

## My Comments and Notes

#### Assessment

## Indeterminate Equation



Practice 1

Find all ordered integer pairs (x, y) such that x + xy + y = 8.

Practice 2

Solve in integers the equation 41x + 37y = 13.

Practice 3

Solve in positive integers the following equations:

(i)  $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ (ii)  $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ (iii)  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$ 

Practice 4

Solve the equation  $x^2 + y^2 = 6x - 4y - 13$ .

## Practice 5

How many ordered integer pairs (x, y) are there such that  $5(x^2 + 3) = y^2$ ?

Practice 6

Find all the right triangles that satisfy the following two conditions:

- (i) the lengths of all its three sides are integers, and
- (ii) its area and perimeter are numerically equal

#### Assessment

## Indeterminate Equation



Practice 7

Solve in positive integers the equation  $y^2 = x^2 + x + 1$ .

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Practice 9

Solve in integers the equation:  $x^2 + y^2 = 2015$ .

### Practice 10

Find all positive integer triplets (x, y, z) such that  $3^x + 4^y = 5^z$ .

Practice 11

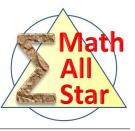
Solve in integers the equation  $x^3 + 2y^3 = 4z^3$ .

### Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

## **Reference Solutions**

## Indeterminate Equation



## Practice 1

Find all ordered integer pairs (x, y) such that x + xy + y = 8.

 $\dot{\gamma}$  Tip: The factorization method and integer divisibility.

Solution 1: Polynomial Factorization

The given equation can be re-written as:

$$(x+1)(y+1) = 9$$

Because both x and y are integers, both (x + 1) and (y + 1) are integers too. It follows that they must be paired divisors of 9:

| x + 1 | y+1 | x   | y   |
|-------|-----|-----|-----|
| 1     | 9   | 0   | 8   |
| 3     | 3   | 2   | 2   |
| 9     | 1   | 8   | 0   |
| -1    | -9  | -2  | -10 |
| -3    | -3  | -4  | -4  |
| -9    | -1  | -10 | -2  |

Therefore we conclude there are totally 6 solutions.

### Solution 2: Integer Divisibility

Rearrange the given equation as an equation with respect to x: (y+1)x = 8 - y. Hence:

$$x = \frac{8 - y}{y + 1} = \frac{9 - (1 + y)}{y + 1} = \frac{9}{y + 1} - 1 \tag{1}$$

Because x is an integer,  $\frac{9}{y+1}$  must be an integer. This follows that (y+1) must be a divisor of 9, or

$$y + 1 = -9, -3, -1, 1, 3, 9$$
  
 $y = -10, -4, -2, 0, 2, 8$ 

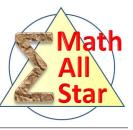
Setting these values to Equation 1, respectively, leads to

$$x = -2, -4, -10, 8, 2, 0$$

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#### Assessment

## Indeterminate Equation



## Practice 2

Solve in integers the equation 41x + 37y = 13.

 $\dot{\mathbf{y}}$  Tip: The Euclidean method, and the ax + by = 1 and ax + by = c patterns.

First, let's solve

$$41x + 37y = 1 \tag{2}$$

This can be done by the Euclidean method. We have

$$41 = 37 \times 1 + 4$$
$$37 = 4 \times 9 + 1$$

Therefore

$$1 = 37 - 4 \times 9 = 37 - (41 - 37 \times 1) \times 9 = -41 \times 9 + 37 \times 10$$

This means Equation 2 has one solution (-9, 10), and its general solution is given by:

$$\begin{cases} x = -9 + 37t \\ y = 10 - 41t \end{cases}$$
(3)

where t is an integer parameter.

It follows that

$$\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases}$$
(4)

is the solution to the original question 41x + 37y = 13.

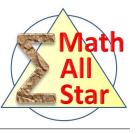
### Practice 3

Solve in positive integers the following equations:

(i)  $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ (ii)  $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ (iii)  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$ 

#### Assessment

## Indeterminate Equation



(i)

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*Tip:* The factorization method, and the 
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$
 pattern.

The given equation can be rewritten as (x-3)(y-3) = 9. Therefore both (x-3) and (y-3) must be divisors of 16. Without loss of generality, let's assume  $x-3 \ge y-3$ . This follows that one of the following must hold:

$$\begin{cases} x - 3 &= 9 \\ y - 3 &= 1 \end{cases} \quad \text{or} \quad \begin{cases} x - 3 &= 3 \\ y - 3 &= 3 \end{cases}$$

Solving these two systems lead to (x, y) = (12, 4), (6, 6). Hence all the solutions are

$$(x, y) = (12, 4), (6, 6), (4, 12)$$

 $\frac{1}{x} \ge \frac{1}{y}$ 

(ii)

$$\dot{v}$$
. Tip: The inequality method, and the  $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$  pattern.

By symmetry, let's assume  $x \leq y$ . Hence

It follows that,

$$\frac{1}{x} \ge \frac{1}{2} \times \frac{5}{6} = \frac{5}{12} \tag{5}$$

or  $x \leq 2$ .

Testing x = 1, 2 respectively finds (2, 3) is one solution. Therefore all the solutions to the given equations are

$$(x, y) = (2, 3), (3, 2)$$

Quiz: Can you use this method to solve (i) above?

(iii)

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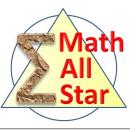
$$\dot{V}$$
 Tip: The inequality method, and the  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{m}{n}$  pattern.

By the symmetrical argument, let's assume  $0 < x \le y \le z$ . It follows:

$$\frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{3}{x}$$

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# Indeterminate Equation



Then 
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5} \implies \frac{1}{x} < \frac{3}{5} \le \frac{3}{x} \implies 2 \le x \le 5.$$

Now we proceed with casework:

If x = 2, then  $\frac{1}{u} + \frac{1}{z} = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$ . If x = 3, then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}$ . If x = 4, then  $\frac{1}{u} + \frac{1}{z} = \frac{3}{5} - \frac{1}{4} = \frac{7}{20}$ . If x = 5, then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$ .

The equivalent equation in every case is in the form of:

 $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$  or  $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$ 

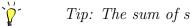
They can all be solved by using the techniques that are presented in (i) and (ii) above. Solving these equations leads to the following solutions under the assumption  $0 < x \leq y \leq z$ .

(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 5), (5, 10), (5, 110), and (5, 5, 5).

Therefore, all the solutions are just distinct permutations of the above set.

Practice 4

Solve the equation  $x^2 + y^2 = 6x - 4y - 13$ .



Tip: The sum of square method.

The given equation is equivalent to:

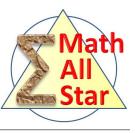
$$(x-3)^2 + (y+2)^2 = 0$$

Because squares cannot be negative, the only possibility to make this equation hold is (x, y) = (3, -2).

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## Indeterminate Equation



## Practice 5

How many ordered integer pairs (x, y) are there such that  $5(x^2 + 3) = y^2$ ?



Tip: Property of square numbers.

It is obvious that  $(x^2 + 3)$  must be a multiple of 5. It follows that the unit digit of  $(x^2 + 3)$  must be either 0 or 5. Equivalently,  $x^2$  must end with 8 or 3. However no square number can end with 8 or 3. Hence this equation is not solvable.

#### Practice 6

Find all the right triangles that satisfy the following two conditions:

- (i) the lengths of all its three sides are integers, and
- (ii) its area and perimeter are numerically equal

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Tip: The Pythagorean triplet formula.

By the Pythagorean triplet formula, the lengths of three sides can be written as:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}$$

where m and n are two positive integers.

If its area and perimeter equal in values, the following must hold:

$$\frac{1}{2}(m^2 - n^2)(2mn) = (m^2 - n^2) + 2mn + (m^2 + n^2)$$

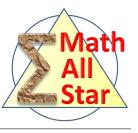
It follows that:

$$(m^2 - n^2)(mn) = 2m^2 + 2mn$$
  
 $(m+n)(m-n)mn = 2m(m+n)$   
 $(m-n)n = 2$ 

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## Indeterminate Equation



This is a basic indeterminate equation that can be solved using the factorization method.

$$\begin{cases} m-n &= 1 \\ n &= 2 \end{cases} \quad \text{or} \quad \begin{cases} m-n &= 2 \\ n &= 1 \end{cases}$$

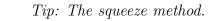
We find (m, n) = (3, 2) or (3, 1).

Setting these values into the Pythagorean triplet formula produces two such triangles: 5-12-13 and 6-8-10.

### Practice 7

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Solve in positive integers the equation  $y^2 = x^2 + x + 1$ .



From the given equation and the condition x > 0, it is easy to see that

$$y^2 < x^2 < (x+1)^2$$

Note that x and (x + 1) are two consecutive integers. Therefore it is impossible to have another integer whose square is between  $x^2$  and  $(x + 1)^2$ .

Hence, we conclude that no solution is possible.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

 $\frac{x}{y} + \frac{15y}{4x}$ 

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Tip: The quadratic method.

Let  $u = \frac{x}{y}$ , then the given problem is equivalent to:

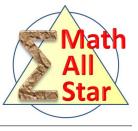
$$u + \frac{15}{4u} = k$$

where k is an integer. It is obvious that u is a positive rational number because both x and y are positive integers.

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## Indeterminate Equation



Rewriting this relationship leads to:

$$4u^2 - 4ku + 15 = 0 \tag{6}$$

Because Equation 6 is solvable in rational number, its discriminant must be a square number. Let

$$\Delta = 16k^2 - 4 \times 4 \times 15 = n^2$$

where n is an integer. Or:

$$16k^2 - n^2 = 240$$

Clearly,  $n^2$  is a multiple of  $16 = 4^2$ , setting n = 4m leads to:

$$16k^{2} - 16m^{2} = 240$$

$$k^{2} - m^{2} = 15$$

$$(k+m)(k-m) = 15$$
(7)

Equation 7 can be solved by the factorization method. Because both k and m are positive integers, we have k + m > k - m. Consequently, one of the two systems must hold:

$$\begin{cases} k+m &= 15\\ k-m &= 1 \end{cases} \quad \text{or} \quad \begin{cases} k+m &= 5\\ k-m &= 3 \end{cases}$$

Solving the above two systems leads to

$$(k,m) = (8,7), (4,1)$$

Setting k = 8 to the quadratic formula of *Equation 6*:

$$u = \frac{4 \times 8 \pm \sqrt{(4 \times 8)^2 - 4 \times 4 \times 15}}{2 \times 4} = 4 \pm \frac{7}{2} = \frac{15}{2}, \frac{1}{2}$$

Setting k = 4 leads:

$$u = \frac{4 \times 4 \pm \sqrt{(4 \times 4)^2 - 4 \times 4 \times 15}}{2 \times 4} = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}$$

Therefore, we conclude there are four solutions:

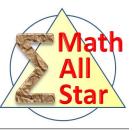
$$(x, y) = (15, 2), (1, 2), (5, 2)$$
 and  $(3, 2)$ 

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## Indeterminate Equation



## Practice 9

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Solve in integers the equation:  $x^2 + y^2 = 2015$ .

Tip: Number theory / Frequently used MOD conclusions.

Taking MOD 4 on both sides leads to

$$x^2 + y^2 = 2015 \equiv 3 \pmod{4}$$

However this relationship cannot hold. Therefore the original equation is not solvable in integers.

### Practice 10

Find all positive integer triplets (x, y, z) such that  $3^x + 4^y = 5^z$ .

## $\dot{\mathbf{v}}$ Tip: The MOD method

First, let's show x, y, and z must be all even.

Taking (mod 4) on both sides of the equation leads to:

 $(-1)^x + 0 \equiv 1^z \pmod{4}$ 

Clearly, this relationship can only hold if x is even.

Next, taking (mod 3) on both sides yields:

$$0 + 1^y \equiv (-1)^z \pmod{3}$$

Therefore, z must be even too.

As such, let x = 2k, z = 2p, and note  $4^y = (2^y)^2$ , the original equation becomes:

$$(3^k)^2 + (2^y)^2 = (5^p)^2$$

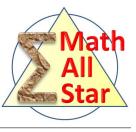
Therefore  $(3^k, 2^y, 5^p)$  forms a Pythagorean triplet. Hence, there exist positive integers m and n such that <sup>1</sup>:

$$\begin{cases} 3^k = m^2 - n^2 \\ 2^y = 2mn \\ 5^p = m^2 + n^2 \end{cases}$$

<sup>&</sup>lt;sup>1</sup>Note  $3^k$  is an odd number. Therefore it cannot equal 2mn

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Because  $2^y = 2mn$ , both m and n must be some power of 2. Let  $m = 2^t$  and  $n = 2^s$  where t and s are non-negative integers satisfying t + s = y - 1. Note  $m > n \implies t > s$ .

It follows that:

 $\begin{cases} 3^k = m^2 - n^2 = 2^{2t} - 2^{2s} = 2^{2s}(2^{2(t-s)} - 1) \\ 5^p = m^2 + n^2 = 2^{2t} + 2^{2s} = 2^{2s}(2^{2(t-s)} + 1) \end{cases}$ 

Because neither  $3^k$  nor  $5^p$  is divisible by 2, we conclude  $2^{2s}$  must equal 1. This means s = 0, and  $2^{2(t-s)} = 4$  or t = 1. It is followed by k = p = 1.

Hence, the given equation has only one positive integer solution: x = y = z = 2.

#### Practice 11

`<mark>?</mark>

Solve in integers the equation  $x^3 + 2y^3 = 4z^3$ .

Tip: The infinite descent method.

If there exists such a positive integer solution (x, y, z), then x must be even. Let  $x = 2x_1$ :

$$(2x_1)^3 + 2y^3 = 4z^3$$
$$4x_1^3 + y^3 = 2z^3$$

This means y must be even too. Let  $y = 2y_1$ :

$$4x_1^3 + (2y_1)^3 = 2z^3$$
  
$$2x_1^3 + 4y_1^3 = z^3$$

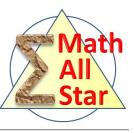
This in turn shows z is also even. Let be  $z = 2z_1$ :

$$2x_1^3 + 4y_1^3 = (2z_1)^3$$
$$x_1^3 + 2y_1^3 = 4z_1^3$$

This last equation is in the same form of the original one. Hence, we conclude if (x, y, z) is a positive integer solution, x, y, and z must be all even, and  $(x_1, y_1, z_1) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$  will be a solution too. It is clear that the process of  $(x, y, z) \implies (x_1, y_1, z_1)$  is repeatable. Therefore an infinitive decreasing solution series can be constructed, which is impossible by the principle of infinite descent. Thus, the equation is unsolvable in positive integers.

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## Indeterminate Equation



## Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

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## $Tip:\ The\ Pell's\ equation.$

Let three sides' lengths be z - 1, z, and z + 1, respectively. Then by the Heron's formula, the triangle's area is given by:

$$S = \sqrt{\frac{3}{2}z \times \left(\frac{3}{2}z - (z-1)\right)\left(\frac{3}{2}z - z\right)\left(\frac{3}{2}z - (z+1)\right)}$$
$$= \frac{z}{4}\sqrt{3(z^2 - 4)}$$
(8)

If S is an integer, then  $3(z^2 - 4)$  must be a square number. Let

$$3(z^2 - 4) = 3w^2$$

In addition, from Equation 8, it is clear that z must be even because, otherwise, both z and  $\sqrt{3(z^2-4)}$  will be odd. This will make S a non-integer.

Letting z = 2x leads to  $4x^2 - 4 = 3w^2$ . This means that w must be even too. Letting w = 2y and simplifying yield:

$$x^2 - 3y^2 = 1$$

This is a Pell's equation. Its fundamental solution is

$$(x,y) = (2,1)$$

and general solution is given by:

$$\begin{cases} x_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2} \\ y_n = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} \end{cases}$$
(9)

As a result, there exist infinitely many such triangles. Three sides are  $(2x_n - 1, 2x_n, 2x_n + 1)$  where  $x_n$  is given by Equation 9, and area is  $3 \cdot x_n \cdot y_n$ .

The three smallest such triangles can be obtained by setting n = 1, 2, and 3, respectively:

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# Indeterminate Equation



## Answer Keys

Practice 1: (x, y) = (0, 8), (8, 0), (2, 2), (-2, -10), (-10, -2), (-4, -4)

Practice 2:

 $\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases}$ 

where t is an integer parameter.

Practice 3:

- (i) (x, y) = (2, 3), (3, 2)
- (ii) (x, y) = (12, 4), (6, 6), (4, 12)
- (iii) Permutation of the following sets: (2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

| Practice 4:  | (x,y) = (3,-2)   |
|--------------|--|
| Practice 5:  | No solution exists.  |
| Practice 6:  | 5-12-13 and 6-8-10.  |
| Practice 7:  | No solution exits.   |
| Practice 8:  | (x, y) = (15, 2), (1, 2), (5, 2) and $(3, 2)$                                |
| Practice 9:  | No solution exits.   |
| Practice 10: | (x, y, z) = (2, 2, 2)  |
| Practice 11: | No solution exits.   |
| Practice 12: | There exist infinitely many such triangles. Refer to the reference solution. |

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