Indeterminate Equation

Assessment

Math for Gifted Students

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Assessment

Indeterminate Equation

Instructions

- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.* C_{100}^{50} *is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good one to practise?*
- *Complete the My Record section below before submission.*

My Comments and Notes

Indeterminate Equation

Practice 1

Find all ordered integer pairs (x, y) such that $x + xy + y = 8$.

Practice 2

Solve in integers the equation $41x + 37y = 13$.

Practice 3

Solve in positive integers the following equations:

(i) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$

(ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

(iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

Practice 4

Solve the equation $x^2 + y^2 = 6x - 4y - 13$.

Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?

Practice 6

Find all the right triangles that satisfy the following two conditions:

(i) the lengths of all its three sides are integers, and

(ii) its area and perimeter are numerically equal

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Practice 7

Solve in positive integers the equation $y^2 = x^2 + x + 1$.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Practice 9

Solve in integers the equation: $x^2 + y^2 = 2015$.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

Practice 11

Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

Reference Solutions

Assessment

Indeterminate Equation

Practice 1

Find all ordered integer pairs (x, y) such that $x + xy + y = 8$.

Tip: The factorization method and integer divisibility.

Solution 1: Polynomial Factorization

The given equation can be re-written as:

$$(x+1)(y+1) = 9$$

Because both x and y are integers, both $(x+1)$ and $(y+1)$ are integers too. It follows that they must be paired divisors of 9:

$x+1$	$y+1$	x	y
1	9	0	8
3	3	2	2
9	1	8	0
-1	-9	-2	-10
-3	-3	-4	-4
-9	-1	-10	-2

Therefore we conclude there are totally 6 solutions.

Solution 2: Integer Divisibility

Rearrange the given equation as an equation with respect to x: $(y+1)x = 8 - y$. Hence:

$$x = \frac{8-y}{y+1} = \frac{9-(1+y)}{y+1} = \frac{9}{y+1} - 1 \tag{1}$$

Because x is an integer, $\dfrac{9}{y+1}$ must be an integer. This follows that $(y+1)$ must be a divisor of 9, or

$$\begin{aligned} y+1 &= -9, -3, -1, 1, 3, 9 \\ y &= -10, -4, -2, 0, 2, 8 \end{aligned}$$

Setting these values to *Equation 1*, respectively, leads to

$$x = -2, -4, -10, 8, 2, 0$$

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Practice 2

Solve in integers the equation $41x + 37y = 13$.

Tip: The Euclidean method, and the $ax + by = 1$ and $ax + by = c$ patterns.

First, let's solve

$$41x + 37y = 1 \tag{2}$$

This can be done by the Euclidean method. We have

$$\begin{aligned} 41 &= 37 \times 1 + 4 \\ 37 &= 4 \times 9 + 1 \end{aligned}$$

Therefore

$$1 = 37 - 4 \times 9 = 37 - (41 - 37 \times 1) \times 9 = -41 \times 9 + 37 \times 10$$

This means *Equation 2* has one solution $(-9, 10)$, and its general solution is given by:

$$\begin{cases} x &= -9 + 37t \\ y &= 10 - 41t \end{cases} \tag{3}$$

where t is an integer parameter.

It follows that

$$\begin{cases} x &= -9 \times 13 + 37 \times 13t &= -117 + 481t \\ y &= 10 \times 13 - 41 \times 13t &= 130 - 533t \end{cases} \tag{4}$$

is the solution to the original question $41x + 37y = 13$.

Practice 3

Solve in positive integers the following equations:

(i) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$

(ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

(iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

Indeterminate Equation

(i)

Tip: The factorization method, and the $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ *pattern.*

The given equation can be rewritten as $(x-3)(y-3) = 9$. Therefore both $(x-3)$ and $(y-3)$ must be divisors of 16. Without loss of generality, let's assume $x - 3 \geq y - 3$. This follows that one of the following must hold:

$$\begin{cases} x-3 &= 9 \\ y-3 &= 1 \end{cases} \quad \text{or} \quad \begin{cases} x-3 &= 3 \\ y-3 &= 3 \end{cases}$$

Solving these two systems lead to $(x, y) = (12, 4), (6, 6)$. Hence all the solutions are

$$(x, y) = (12, 4), (6, 6), (4, 12)$$

(ii)

Tip: The inequality method, and the $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$ *pattern.*

By symmetry, let's assume $x \leq y$. Hence

$$\frac{1}{x} \geq \frac{1}{y}$$

It follows that,

$$\frac{1}{x} \geq \frac{1}{2} \times \frac{5}{6} = \frac{5}{12} \tag{5}$$

or $x \leq 2$.

Testing $x = 1, 2$ respectively finds $(2, 3)$ is one solution. Therefore all the solutions to the given equations are

$$(x, y) = (2, 3), (3, 2)$$

Quiz: Can you use this method to solve (i) above?

(iii)

Tip: The inequality method, and the $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{m}{n}$ *pattern.*

By the symmetrical argument, let's assume $0 < x \leq y \leq z$. It follows:

$$\frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{x}$$

Then $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{5} \implies \dfrac{1}{x}<\dfrac{3}{5}\le\dfrac{3}{x} \implies 2\le x\le 5.$

Now we proceed with casework:

If $x=2$, then $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{5}-\dfrac{1}{2}=\dfrac{1}{10}$.

If $x=3$, then $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{5}-\dfrac{1}{3}=\dfrac{4}{15}$.

If $x=4$, then $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{5}-\dfrac{1}{4}=\dfrac{7}{20}$.

If $x=5$, then $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{5}-\dfrac{1}{5}=\dfrac{2}{5}$.

The equivalent equation in every case is in the form of:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n} \quad \text{or} \quad \frac{1}{x}+\frac{1}{y}=\frac{m}{n}$$

They can all be solved by using the techniques that are presented in (i) and (ii) above. Solving these equations leads to the following solutions under the assumption $0<x\le y\le z$.

(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Therefore, all the solutions are just distinct permutations of the above set.

Practice 4

Solve the equation $x^2+y^2=6x-4y-13$.

Tip: The sum of square method.

The given equation is equivalent to:

$$(x-3)^2+(y+2)^2=0$$

Because squares cannot be negative, the only possibility to make this equation hold is $(x,y)=(3,-2)$.

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Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?

Tip: Property of square numbers.

It is obvious that $(x^2 + 3)$ must be a multiple of 5. It follows that the unit digit of $(x^2 + 3)$ must be either 0 or 5. Equivalently, x^2 must end with 8 or 3. However no square number can end with 8 or 3. Hence this equation is not solvable.

Practice 6

Find all the right triangles that satisfy the following two conditions:

(i) the lengths of all its three sides are integers, and

(ii) its area and perimeter are numerically equal

Tip: The Pythagorean triplet formula.

By the Pythagorean triplet formula, the lengths of three sides can be written as:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}$$

where m and n are two positive integers.

If its area and perimeter equal in values, the following must hold:

$$\frac{1}{2}(m^2 - n^2)(2mn) = (m^2 - n^2) + 2mn + (m^2 + n^2)$$

It follows that:

$$\begin{aligned} (m^2 - n^2)(mn) &= 2m^2 + 2mn \\ (m+n)(m-n)mn &= 2m(m+n) \\ (m-n)n &= 2 \end{aligned}$$

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This is a basic indeterminate equation that can be solved using the factorization method.

$$\begin{cases} m-n &= 1 \\ n &= 2 \end{cases} \quad \text{or} \quad \begin{cases} m-n &= 2 \\ n &= 1 \end{cases}$$

We find $(m, n) = (3, 2)$ or $(3, 1)$.

Setting these values into the Pythagorean triplet formula produces two such triangles: 5-12-13 and 6-8-10.

Practice 7

Solve in positive integers the equation $y^2 = x^2 + x + 1$.

Tip: The squeeze method.

From the given equation and the condition $x > 0$, it is easy to see that

$$y^2 < x^2 < (x+1)^2$$

Note that x and $(x+1)$ are two consecutive integers. Therefore it is impossible to have another integer whose square is between x^2 and $(x+1)^2$.

Hence, we conclude that no solution is possible.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Tip: The quadratic method.

Let $u = \frac{x}{y}$, then the given problem is equivalent to:

$$u + \frac{15}{4u} = k$$

where k is an integer. It is obvious that u is a positive rational number because both x and y are positive integers.

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Rewriting this relationship leads to:

$$4u^2 - 4ku + 15 = 0 \tag{6}$$

Because *Equation 6* is solvable in rational number, its discriminant must be a square number. Let

$$\Delta = 16k^2 - 4 \times 4 \times 15 = n^2$$

where n is an integer. Or:

$$16k^2 - n^2 = 240$$

Clearly, n^2 is a multiple of $16 = 4^2$, setting $n = 4m$ leads to:

$$\begin{aligned} 16k^2 - 16m^2 &= 240 \\ k^2 - m^2 &= 15 \\ (k+m)(k-m) &= 15 \end{aligned} \tag{7}$$

Equation 7 can be solved by the factorization method. Because both k and m are positive integers, we have $k + m > k - m$. Consequently, one of the two systems must hold:

$$\begin{cases} k+m &= 15 \\ k-m &= 1 \end{cases} \quad \text{or} \quad \begin{cases} k+m &= 5 \\ k-m &= 3 \end{cases}$$

Solving the above two systems leads to

$$(k, m) = (8, 7), (4, 1)$$

Setting $k = 8$ to the quadratic formula of *Equation 6*:

$$u = \frac{4 \times 8 \pm \sqrt{(4 \times 8)^2 - 4 \times 4 \times 15}}{2 \times 4} = 4 \pm \frac{7}{2} = \frac{15}{2}, \frac{1}{2}$$

Setting $k = 4$ leads:

$$u = \frac{4 \times 4 \pm \sqrt{(4 \times 4)^2 - 4 \times 4 \times 15}}{2 \times 4} = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}$$

Therefore, we conclude there are four solutions:

$$(x, y) = (15, 2), (1, 2), (5, 2) \text{ and } (3, 2)$$

Indeterminate Equation

Practice 9

Solve in integers the equation: $x^2 + y^2 = 2015$.

Tip: Number theory / Frequently used MOD conclusions.

Taking MOD 4 on both sides leads to

$$x^2 + y^2 = 2015 \equiv 3 \pmod 4$$

However this relationship cannot hold. Therefore the original equation is not solvable in integers.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

Tip: The MOD method

First, let's show x, y, and z must be all even.

Taking $\pmod 4$ on both sides of the equation leads to:

$$(-1)^x + 0 \equiv 1^z \pmod 4$$

Clearly, this relationship can only hold if x is even.

Next, taking $\pmod 3$ on both sides yields:

$$0 + 1^y \equiv (-1)^z \pmod 3$$

Therefore, z must be even too.

As such, let $x = 2k$, $z = 2p$, and note $4^y = (2^y)^2$, the original equation becomes:

$$(3^k)^2 + (2^y)^2 = (5^p)^2$$

Therefore $(3^k, 2^y, 5^p)$ forms a Pythagorean triplet. Hence, there exist positive integers m and n such that [1]:

$$\begin{cases} 3^k &= m^2 - n^2 \\ 2^y &= 2mn \\ 5^p &= m^2 + n^2 \end{cases}$$

[1] Note 3^k is an odd number. Therefore it cannot equal $2mn$

Because $2^y = 2mn$, both m and n must be some power of 2. Let $m = 2^t$ and $n = 2^s$ where t and s are non-negative integers satisfying $t + s = y - 1$. Note $m > n \implies t > s$.

It follows that:

$$\begin{cases} 3^k &= m^2 - n^2 &= 2^{2t} - 2^{2s} &= 2^{2s}(2^{2(t-s)} - 1) \\ 5^p &= m^2 + n^2 &= 2^{2t} + 2^{2s} &= 2^{2s}(2^{2(t-s)} + 1) \end{cases}$$

Because neither 3^k nor 5^p is divisible by 2, we conclude 2^{2s} must equal 1. This means $s = 0$, and $2^{2(t-s)} = 4$ or $t = 1$. It is followed by $k = p = 1$.

Hence, the given equation has only one positive integer solution: $x = y = z = 2$.

Practice 11

Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Tip: The infinite descent method.

If there exists such a positive integer solution (x, y, z), then x must be even. Let $x = 2x_1$:

$$\begin{aligned} (2x_1)^3 + 2y^3 &= 4z^3 \\ 4x_1^3 + y^3 &= 2z^3 \end{aligned}$$

This means y must be even too. Let $y = 2y_1$:

$$\begin{aligned} 4x_1^3 + (2y_1)^3 &= 2z^3 \\ 2x_1^3 + 4y_1^3 &= z^3 \end{aligned}$$

This in turn shows z is also even. Let be $z = 2z_1$:

$$\begin{aligned} 2x_1^3 + 4y_1^3 &= (2z_1)^3 \\ x_1^3 + 2y_1^3 &= 4z_1^3 \end{aligned}$$

This last equation is in the same form of the original one. Hence, we conclude if (x, y, z) is a positive integer solution, x, y, and z must be all even, and $(x_1, y_1, z_1) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ will be a solution too. It is clear that the process of $(x, y, z) \implies (x_1, y_1, z_1)$ is repeatable. Therefore an infinitive decreasing solution series can be constructed, which is impossible by the principle of infinite descent. Thus, the equation is unsolvable in positive integers.

Indeterminate Equation

Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

Tip: The Pell's equation.

Let three sides' lengths be $z-1$, z, and $z+1$, respectively. Then by the Heron's formula, the triangle's area is given by:

$$S = \sqrt{\frac{3}{2}z \times \left(\frac{3}{2}z-(z-1)\right)\left(\frac{3}{2}z-z\right)\left(\frac{3}{2}z-(z+1)\right)}$$

$$= \frac{z}{4}\sqrt{3(z^2-4)} \tag{8}$$

If S is an integer, then $3(z^2-4)$ must be a square number. Let

$$3(z^2-4) = 3w^2$$

In addition, from *Equation 8*, it is clear that z must be even because, otherwise, both z and $\sqrt{3(z^2-4)}$ will be odd. This will make S a non-integer.

Letting $z=2x$ leads to $4x^2-4=3w^2$. This means that w must be even too. Letting $w=2y$ and simplifying yield:

$$x^2-3y^2=1$$

This is a Pell's equation. Its fundamental solution is

$$(x,y)=(2,1)$$

and general solution is given by:

$$\begin{cases} x_n = \dfrac{(2+\sqrt{3})^n+(2-\sqrt{3})^n}{2} \\ \\ y_n = \dfrac{(2+\sqrt{3})^n-(2-\sqrt{3})^n}{2\sqrt{3}} \end{cases} \tag{9}$$

As a result, there exist infinitely many such triangles. Three sides are $(2x_n-1, 2x_n, , 2x_n+1)$ where x_n is given by *Equation 9*, and area is $3 \cdot x_n \cdot y_n$.

The three smallest such triangles can be obtained by setting $n = 1, 2$, and 3, respectively:

$$(3,4,5), (13,14,15), (51,52,53)$$

Answer Keys

Practice 1: $(x, y) = (0, 8), (8, 0), (2, 2), (-2, -10), (-10, -2), (-4, -4)$

Practice 2:

$$\begin{cases} x &= -9 \times 13 + 37 \times 13t &= -117 + 481t \\ y &= 10 \times 13 - 41 \times 13t &= 130 - 533t \end{cases}$$

where t is an integer parameter.

Practice 3:

(i) $(x, y) = (2, 3), (3, 2)$

(ii) $(x, y) = (12, 4), (6, 6), (4, 12)$

(iii) Permutation of the following sets: (2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Practice 4: $(x, y) = (3, -2)$

Practice 5: No solution exists.

Practice 6: 5-12-13 and 6-8-10.

Practice 7: No solution exits.

Practice 8: $(x, y) = (15, 2), (1, 2), (5, 2)$ and $(3, 2)$

Practice 9: No solution exits.

Practice 10: $(x, y, z) = (2, 2, 2)$

Practice 11: No solution exits.

Practice 12: There exist infinitely many such triangles. Refer to the reference solution.