Indeterminate Equation

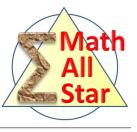
Assessment



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Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g. C_{100}^{50} is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good one to practise?
- Complete the My Record section below before submission.

My Comments and Notes

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Practice 1

Find all ordered integer pairs (x, y) such that x + xy + y = 8.

Practice 2

Solve in integers the equation 41x + 37y = 13.

Practice 3

Solve in positive integers the following equations:

(i) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ (ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ (iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

Practice 4

Solve the equation $x^2 + y^2 = 6x - 4y - 13$.

Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?

Practice 6

Find all the right triangles that satisfy the following two conditions:

- (i) the lengths of all its three sides are integers, and
- (ii) its area and perimeter are numerically equal

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Practice 7

Solve in positive integers the equation $y^2 = x^2 + x + 1$.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

Practice 9

Solve in integers the equation: $x^2 + y^2 = 2015$.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

Practice 11

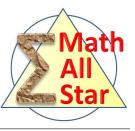
Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

Reference Solutions

Indeterminate Equation



Practice 1

Find all ordered integer pairs (x, y) such that x + xy + y = 8.

 $\dot{\gamma}$ Tip: The factorization method and integer divisibility.

Solution 1: Polynomial Factorization

The given equation can be re-written as:

$$(x+1)(y+1) = 9$$

Because both x and y are integers, both (x + 1) and (y + 1) are integers too. It follows that they must be paired divisors of 9:

x + 1	y+1	x	y
1	9	0	8
3	3	2	2
9	1	8	0
-1	-9	-2	-10
-3	-3	-4	-4
-9	-1	-10	-2

Therefore we conclude there are totally 6 solutions.

Solution 2: Integer Divisibility

Rearrange the given equation as an equation with respect to x: (y+1)x = 8 - y. Hence:

$$x = \frac{8 - y}{y + 1} = \frac{9 - (1 + y)}{y + 1} = \frac{9}{y + 1} - 1 \tag{1}$$

Because x is an integer, $\frac{9}{y+1}$ must be an integer. This follows that (y+1) must be a divisor of 9, or

$$y + 1 = -9, -3, -1, 1, 3, 9$$

 $y = -10, -4, -2, 0, 2, 8$

Setting these values to Equation 1, respectively, leads to

$$x = -2, -4, -10, 8, 2, 0$$

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Practice 2

Solve in integers the equation 41x + 37y = 13.

 $\dot{\mathbf{y}}$ Tip: The Euclidean method, and the ax + by = 1 and ax + by = c patterns.

First, let's solve

$$41x + 37y = 1 \tag{2}$$

This can be done by the Euclidean method. We have

$$41 = 37 \times 1 + 4$$
$$37 = 4 \times 9 + 1$$

Therefore

$$1 = 37 - 4 \times 9 = 37 - (41 - 37 \times 1) \times 9 = -41 \times 9 + 37 \times 10$$

This means Equation 2 has one solution (-9, 10), and its general solution is given by:

$$\begin{cases} x = -9 + 37t \\ y = 10 - 41t \end{cases}$$
(3)

where t is an integer parameter.

It follows that

$$\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases}$$
(4)

is the solution to the original question 41x + 37y = 13.

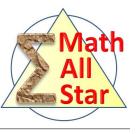
Practice 3

Solve in positive integers the following equations:

(i) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$ (ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ (iii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

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(i)

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Tip: The factorization method, and the
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$
 pattern.

The given equation can be rewritten as (x-3)(y-3) = 9. Therefore both (x-3) and (y-3) must be divisors of 16. Without loss of generality, let's assume $x-3 \ge y-3$. This follows that one of the following must hold:

$$\begin{cases} x - 3 &= 9 \\ y - 3 &= 1 \end{cases} \quad \text{or} \quad \begin{cases} x - 3 &= 3 \\ y - 3 &= 3 \end{cases}$$

Solving these two systems lead to (x, y) = (12, 4), (6, 6). Hence all the solutions are

$$(x, y) = (12, 4), (6, 6), (4, 12)$$

 $\frac{1}{x} \ge \frac{1}{y}$

(ii)

$$\dot{v}$$
. Tip: The inequality method, and the $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$ pattern.

By symmetry, let's assume $x \leq y$. Hence

It follows that,

$$\frac{1}{x} \ge \frac{1}{2} \times \frac{5}{6} = \frac{5}{12} \tag{5}$$

or $x \leq 2$.

Testing x = 1, 2 respectively finds (2, 3) is one solution. Therefore all the solutions to the given equations are

$$(x, y) = (2, 3), (3, 2)$$

Quiz: Can you use this method to solve (i) above?

(iii)

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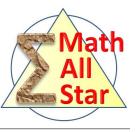
$$\dot{V}$$
 Tip: The inequality method, and the $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{m}{n}$ pattern.

By the symmetrical argument, let's assume $0 < x \le y \le z$. It follows:

$$\frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{3}{x}$$

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Then
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5} \implies \frac{1}{x} < \frac{3}{5} \le \frac{3}{x} \implies 2 \le x \le 5.$$

Now we proceed with casework:

If x = 2, then $\frac{1}{u} + \frac{1}{z} = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$. If x = 3, then $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}$. If x = 4, then $\frac{1}{u} + \frac{1}{z} = \frac{3}{5} - \frac{1}{4} = \frac{7}{20}$. If x = 5, then $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$.

The equivalent equation in every case is in the form of:

 $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ or $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$

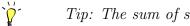
They can all be solved by using the techniques that are presented in (i) and (ii) above. Solving these equations leads to the following solutions under the assumption $0 < x \leq y \leq z$.

(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 5), (5, 10), (5, 110), and (5, 5, 5).

Therefore, all the solutions are just distinct permutations of the above set.

Practice 4

Solve the equation $x^2 + y^2 = 6x - 4y - 13$.



Tip: The sum of square method.

The given equation is equivalent to:

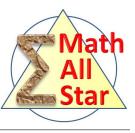
$$(x-3)^2 + (y+2)^2 = 0$$

Because squares cannot be negative, the only possibility to make this equation hold is (x, y) = (3, -2).

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Practice 5

How many ordered integer pairs (x, y) are there such that $5(x^2 + 3) = y^2$?



Tip: Property of square numbers.

It is obvious that $(x^2 + 3)$ must be a multiple of 5. It follows that the unit digit of $(x^2 + 3)$ must be either 0 or 5. Equivalently, x^2 must end with 8 or 3. However no square number can end with 8 or 3. Hence this equation is not solvable.

Practice 6

Find all the right triangles that satisfy the following two conditions:

- (i) the lengths of all its three sides are integers, and
- (ii) its area and perimeter are numerically equal

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Tip: The Pythagorean triplet formula.

By the Pythagorean triplet formula, the lengths of three sides can be written as:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}$$

where m and n are two positive integers.

If its area and perimeter equal in values, the following must hold:

$$\frac{1}{2}(m^2 - n^2)(2mn) = (m^2 - n^2) + 2mn + (m^2 + n^2)$$

It follows that:

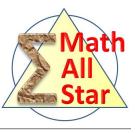
$$(m^2 - n^2)(mn) = 2m^2 + 2mn$$

 $(m+n)(m-n)mn = 2m(m+n)$
 $(m-n)n = 2$

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This is a basic indeterminate equation that can be solved using the factorization method.

$$\begin{cases} m-n &= 1 \\ n &= 2 \end{cases} \quad \text{or} \quad \begin{cases} m-n &= 2 \\ n &= 1 \end{cases}$$

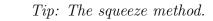
We find (m, n) = (3, 2) or (3, 1).

Setting these values into the Pythagorean triplet formula produces two such triangles: 5-12-13 and 6-8-10.

Practice 7

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Solve in positive integers the equation $y^2 = x^2 + x + 1$.



From the given equation and the condition x > 0, it is easy to see that

$$y^2 < x^2 < (x+1)^2$$

Note that x and (x + 1) are two consecutive integers. Therefore it is impossible to have another integer whose square is between x^2 and $(x + 1)^2$.

Hence, we conclude that no solution is possible.

Practice 8

Find all pairs of positive integers (x, y) where x and y are relatively prime, such that the following expression is an integer:

 $\frac{x}{y} + \frac{15y}{4x}$

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Tip: The quadratic method.

Let $u = \frac{x}{y}$, then the given problem is equivalent to:

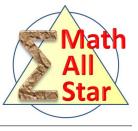
$$u + \frac{15}{4u} = k$$

where k is an integer. It is obvious that u is a positive rational number because both x and y are positive integers.

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Rewriting this relationship leads to:

$$4u^2 - 4ku + 15 = 0 \tag{6}$$

Because Equation 6 is solvable in rational number, its discriminant must be a square number. Let

$$\Delta = 16k^2 - 4 \times 4 \times 15 = n^2$$

where n is an integer. Or:

$$16k^2 - n^2 = 240$$

Clearly, n^2 is a multiple of $16 = 4^2$, setting n = 4m leads to:

$$16k^{2} - 16m^{2} = 240$$

$$k^{2} - m^{2} = 15$$

$$(k+m)(k-m) = 15$$
(7)

Equation 7 can be solved by the factorization method. Because both k and m are positive integers, we have k + m > k - m. Consequently, one of the two systems must hold:

$$\begin{cases} k+m &= 15\\ k-m &= 1 \end{cases} \quad \text{or} \quad \begin{cases} k+m &= 5\\ k-m &= 3 \end{cases}$$

Solving the above two systems leads to

$$(k,m) = (8,7), (4,1)$$

Setting k = 8 to the quadratic formula of *Equation 6*:

$$u = \frac{4 \times 8 \pm \sqrt{(4 \times 8)^2 - 4 \times 4 \times 15}}{2 \times 4} = 4 \pm \frac{7}{2} = \frac{15}{2}, \frac{1}{2}$$

Setting k = 4 leads:

$$u = \frac{4 \times 4 \pm \sqrt{(4 \times 4)^2 - 4 \times 4 \times 15}}{2 \times 4} = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}$$

Therefore, we conclude there are four solutions:

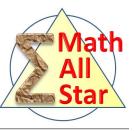
$$(x, y) = (15, 2), (1, 2), (5, 2)$$
 and $(3, 2)$

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Practice 9

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Solve in integers the equation: $x^2 + y^2 = 2015$.

Tip: Number theory / Frequently used MOD conclusions.

Taking MOD 4 on both sides leads to

$$x^2 + y^2 = 2015 \equiv 3 \pmod{4}$$

However this relationship cannot hold. Therefore the original equation is not solvable in integers.

Practice 10

Find all positive integer triplets (x, y, z) such that $3^x + 4^y = 5^z$.

$\dot{\mathbf{v}}$ Tip: The MOD method

First, let's show x, y, and z must be all even.

Taking (mod 4) on both sides of the equation leads to:

 $(-1)^x + 0 \equiv 1^z \pmod{4}$

Clearly, this relationship can only hold if x is even.

Next, taking (mod 3) on both sides yields:

$$0 + 1^y \equiv (-1)^z \pmod{3}$$

Therefore, z must be even too.

As such, let x = 2k, z = 2p, and note $4^y = (2^y)^2$, the original equation becomes:

$$(3^k)^2 + (2^y)^2 = (5^p)^2$$

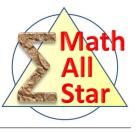
Therefore $(3^k, 2^y, 5^p)$ forms a Pythagorean triplet. Hence, there exist positive integers m and n such that ¹:

$$\begin{cases} 3^k = m^2 - n^2 \\ 2^y = 2mn \\ 5^p = m^2 + n^2 \end{cases}$$

¹Note 3^k is an odd number. Therefore it cannot equal 2mn

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Because $2^y = 2mn$, both m and n must be some power of 2. Let $m = 2^t$ and $n = 2^s$ where t and s are non-negative integers satisfying t + s = y - 1. Note $m > n \implies t > s$.

It follows that:

 $\begin{cases} 3^k = m^2 - n^2 = 2^{2t} - 2^{2s} = 2^{2s}(2^{2(t-s)} - 1) \\ 5^p = m^2 + n^2 = 2^{2t} + 2^{2s} = 2^{2s}(2^{2(t-s)} + 1) \end{cases}$

Because neither 3^k nor 5^p is divisible by 2, we conclude 2^{2s} must equal 1. This means s = 0, and $2^{2(t-s)} = 4$ or t = 1. It is followed by k = p = 1.

Hence, the given equation has only one positive integer solution: x = y = z = 2.

Practice 11

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Solve in integers the equation $x^3 + 2y^3 = 4z^3$.

Tip: The infinite descent method.

If there exists such a positive integer solution (x, y, z), then x must be even. Let $x = 2x_1$:

$$(2x_1)^3 + 2y^3 = 4z^3$$
$$4x_1^3 + y^3 = 2z^3$$

This means y must be even too. Let $y = 2y_1$:

$$4x_1^3 + (2y_1)^3 = 2z^3$$

$$2x_1^3 + 4y_1^3 = z^3$$

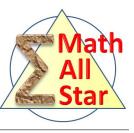
This in turn shows z is also even. Let be $z = 2z_1$:

$$2x_1^3 + 4y_1^3 = (2z_1)^3$$
$$x_1^3 + 2y_1^3 = 4z_1^3$$

This last equation is in the same form of the original one. Hence, we conclude if (x, y, z) is a positive integer solution, x, y, and z must be all even, and $(x_1, y_1, z_1) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$ will be a solution too. It is clear that the process of $(x, y, z) \implies (x_1, y_1, z_1)$ is repeatable. Therefore an infinitive decreasing solution series can be constructed, which is impossible by the principle of infinite descent. Thus, the equation is unsolvable in positive integers.

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Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

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$Tip:\ The\ Pell's\ equation.$

Let three sides' lengths be z - 1, z, and z + 1, respectively. Then by the Heron's formula, the triangle's area is given by:

$$S = \sqrt{\frac{3}{2}z \times \left(\frac{3}{2}z - (z-1)\right)\left(\frac{3}{2}z - z\right)\left(\frac{3}{2}z - (z+1)\right)}$$
$$= \frac{z}{4}\sqrt{3(z^2 - 4)}$$
(8)

If S is an integer, then $3(z^2 - 4)$ must be a square number. Let

$$3(z^2 - 4) = 3w^2$$

In addition, from Equation 8, it is clear that z must be even because, otherwise, both z and $\sqrt{3(z^2-4)}$ will be odd. This will make S a non-integer.

Letting z = 2x leads to $4x^2 - 4 = 3w^2$. This means that w must be even too. Letting w = 2y and simplifying yield:

$$x^2 - 3y^2 = 1$$

This is a Pell's equation. Its fundamental solution is

$$(x,y) = (2,1)$$

and general solution is given by:

$$\begin{cases} x_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2} \\ y_n = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} \end{cases}$$
(9)

As a result, there exist infinitely many such triangles. Three sides are $(2x_n - 1, 2x_n, 2x_n + 1)$ where x_n is given by Equation 9, and area is $3 \cdot x_n \cdot y_n$.

The three smallest such triangles can be obtained by setting n = 1, 2, and 3, respectively:

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Answer Keys

Practice 1: (x, y) = (0, 8), (8, 0), (2, 2), (-2, -10), (-10, -2), (-4, -4)

Practice 2:

 $\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases}$

where t is an integer parameter.

Practice 3:

- (i) (x, y) = (2, 3), (3, 2)
- (ii) (x, y) = (12, 4), (6, 6), (4, 12)
- (iii) Permutation of the following sets: (2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Practice 4:	(x,y) = (3,-2)
Practice 5:	No solution exists.
Practice 6:	5-12-13 and 6-8-10.
Practice 7:	No solution exits.
Practice 8:	(x, y) = (15, 2), (1, 2), (5, 2) and $(3, 2)$
Practice 9:	No solution exits.
Practice 10:	(x, y, z) = (2, 2, 2)
Practice 11:	No solution exits.
Practice 12:	There exist infinitely many such triangles. Refer to the reference solution.

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