Geometry

The Area Method

Math All Star

Math for Gifted Students

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Sample Contents

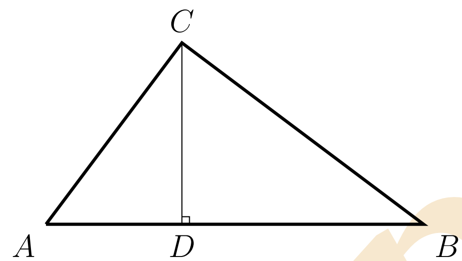
The Area Method

Practice 1

In $\triangle ABC$, $\angle C$ is the right angle. Let D be the foot of the altitude drawn from C on AB. Show that:

$$\overline{CD} = \frac{\overline{AC} \cdot \overline{BC}}{\overline{AB}}$$

This is a classic problem which can be solved using the area method.



Let S be the area of $\triangle ABC$. It can be computed using two different ways:

- method 1: $S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BC}$.
- method 2: $S = \frac{1}{2} \cdot \overline{AB} \cdot \overline{CD}$.

Therefore, we have $\frac{1}{2} \cdot \overline{AC} \cdot \overline{BC} = \frac{1}{2} \cdot \overline{AB} \cdot \overline{CD}$, or

$$\overline{CD} = \frac{\overline{AC} \cdot \overline{BC}}{\overline{AB}}$$

Despite simplicity, this example shows the typical way to employ the area method:

step 1: Identify the target. In this case, it is the area of $\triangle ABC$.

step 2: Find two different ways to compute the identified target.

step 3: Set the two expressions equal and solve for results

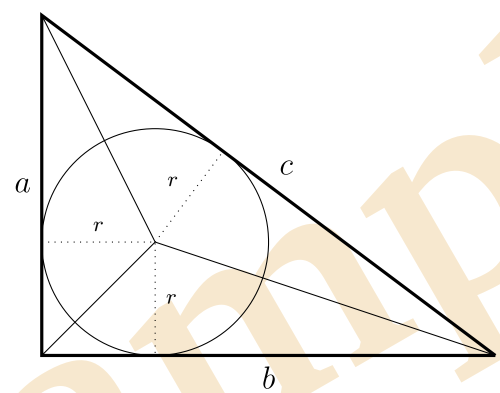
The Area Method

Practice 2

Let a, b, and c be the three sides of a right triangle, and r be the radius of this triangle's inscribed circle. Express r using a, b, and c.

This problem can be solved in several different ways. The area method offers a general solution (please refer to the quiz later).

Let's connect the incenter to all the three vertices, respectively, as shown. It divides the original triangle into three smaller triangles.



It is clear that the area of this original right triangle must equal to the sum of the areas of these three smaller triangles. Hence, we have:

$$\frac{1}{2} \cdot ab = \frac{1}{2} \cdot ar + \frac{1}{2} \cdot br + \frac{1}{2} \cdot cr$$

or

$$r = \frac{ab}{a+b+c} \tag{1}$$

Tip: By the Pythagorean theorem, Equation 3 can be transformed to

$$r = \frac{a+b-c}{2} \tag{2}$$

In the case of a right triangle, *Equation 4* is easier to remember and use than *Equation 3*, though these two are equivalent.

Quiz: Equation 4 can be derived directly using other method as well. Do you want to have a try?

Quiz: Can you use the general formula of r for an arbitrary triangle?

Practice 3

Let $ABCD$ be a tetrahedron, and

- Its base BCD is an equilateral triangle
- All its three sides are right triangles, where $\angle A$ is always the right angle
- $AB = AC = AD = 1$

Find the distance from A to the base BCD.

Tip: It is also possible to apply the concept of the area method to volume.

Tip: This particular problem can also be solved using coordinate geometry. We will revisit it in other practices.

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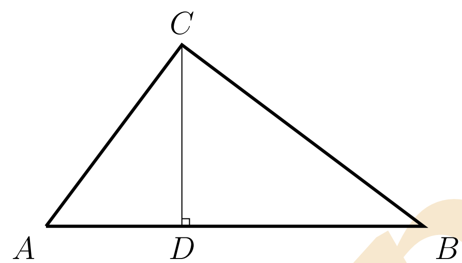
Practice 4

In $\triangle ABC$, $\angle C$ is the right angle. Let D be the foot of the altitude drawn from C on AB. Show that:

$$\overline{CD} = \frac{\overline{AC} \cdot \overline{BC}}{\overline{AB}}$$

This is a classic problem which can be solved using the area method.



Let S be the area of $\triangle ABC$. It can be computed using two different ways:

- method 1: $S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BC}$.
- method 2: $S = \frac{1}{2} \cdot \overline{AB} \cdot \overline{CD}$.

Therefore, we have $\frac{1}{2} \cdot \overline{AC} \cdot \overline{BC} = \frac{1}{2} \cdot \overline{AB} \cdot \overline{CD}$, or

$$\overline{CD} = \frac{\overline{AC} \cdot \overline{BC}}{\overline{AB}}$$

Despite simplicity, this example shows the typical way to employ the area method:

step 1: Identify the target. In this case, it is the area of $\triangle ABC$.

step 2: Find two different ways to compute the identified target.

step 3: Set the two expressions equal and solve for results

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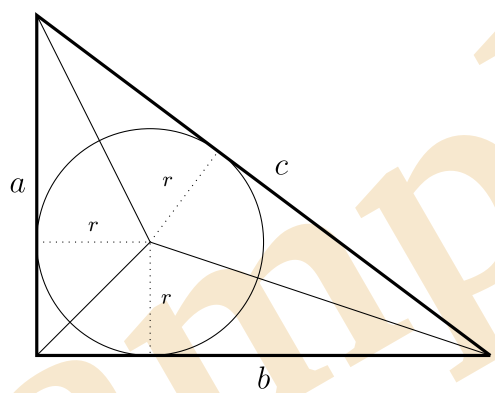
Practice 5

Let a, b, and c be the three sides of a right triangle, and r be the radius of this triangle's inscribed circle. Express r using a, b, and c.

This problem can be solved in several different ways. The area method offers a general solution (please refer to the quiz later).

Let's connect the incenter to all the three vertices, respectively, as shown. It divides the original triangle into three smaller triangles.



It is clear that the area of this original right triangle must equal to the sum of the areas of these three smaller triangles. Hence, we have:

$$\frac{1}{2} \cdot ab = \frac{1}{2} \cdot ar + \frac{1}{2} \cdot br + \frac{1}{2} \cdot cr$$

or

$$r = \frac{ab}{a+b+c} \tag{3}$$

Tip: By the Pythagorean theorem, Equation 3 can be transformed to

$$r = \frac{a+b-c}{2} \tag{4}$$

In the case of a right triangle, *Equation 4* is easier to remember and use than *Equation 3*, though these two are equivalent.

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Quiz: Equation 4 can be derived directly using other method as well. Do you want to have a try?

Quiz: Can you use the general formula of r for an arbitrary triangle?

Practice 6

Let $ABCD$ be a tetrahedron, and

- Its base BCD is an equilateral triangle
- All its three sides are right triangles, where $\angle A$ is always the right angle
- $AB = AC = AD = 1$

Find the distance from A to the base BCD.

Tip: It is also possible to apply the concept of the area method to volume.

Tip: This particular problem can also be solved using coordinate geometry. We will revisit it in other practices.

Battle Field

Here are some related problems selected from recent comptitions:

Problem 1:	2015 MathCounts State Sprint #16
Problem 2:	2012 AMC10B #23
Problem 3:	2002 AMC10A #13
Problem 4:	2015 MathCounts State Sprint #16
Problem 5:	2012 AMC10B #23
Problem 6:	2002 AMC10A #13