Practice

Vieta's Formula (Basic)

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Vieta's Formula (Basic)

Practice 1

Let $x_1, x_2, \cdots, x_n$ be the roots of equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Show that

$$\begin{cases} x_1 + x_2 + \cdots + x_n & = & -a_{n-1} \\ x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots & = & a_{n-2} \\ \cdots & = & \cdots \\ x_1x_2\cdots x_n & = & (-1)^n a_0 \end{cases}$$

Practice 2

Let $x_1, x_2, \cdots, x_n$ be the roots of equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ where $a_n \neq 0$. Describe how Vieta's formula, as shown in the previous practice, will need to change?

Practice 3

If one root of the equation $x^2 - 6x + m^2 - 2m + 5 = 0$ is 2. Find the value of the other root and m.

Practice 4

Let x_1 and x_2 be the two roots of equation $x^2 - 3x + 2 = 0$. Find the following values without computing x_1 and x_2 directly.

i) $x_1^4 + x_2^4$

ii) $x_1 - x_2$

Practice 5

Let x_1 and x_2 be the two roots of $x^2 - 3mx + 2(m-1) = 0$. If $\frac{1}{x_1} + \frac{1}{x_2} = \frac{3}{4}$, what is the value of m?

Practice 6

If the difference of the two roots of the equation $x^2 + 6x + k = 0$ is 2, what is the value of k?

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Practice 7

If the two roots of $x^2 + ax + b + 1 = 0$ are positive integers, show that $a^2 + b^2$ cannot be a prime number.

Practice 8

Find the sum of all possible integer values of a such that the equation

$$(a+1)x^2 - (a^2+1)x + (2a^2 - 6) = 0$$

is solvable in integers.

Practice 9

Let x_1 and x_2 be the two real roots of the equation $x^2 - 2(k+1)x + k^2 + 2 = 0$. If $(x_1+1)(x_2+1) = 8$, find the value of k

Practice 10

Three of the roots of $x^4 + ax^2 + bx + c = 0$ are 2,-3, and 5. Find the value of $a + b + c$.
(Ref 1998 HMMT)

Practice 11

If real number m and n satisfy $mn \neq 1$ and $19m^2 + 99m + 1 = 0$ and $19 + 99n + n^2 = 0$, what is the value of $\frac{mn+4m+1}{n}$?

Reference Solutions

Vieta's Formula (Basic)

Practice 1

Let $x_1, x_2, \cdots, x_n$ be the roots of equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Show that

$$\begin{cases} x_1 + x_2 + \cdots + x_n & = & -a_{n-1} \\ x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots & = & a_{n-2} \\ \cdots & = & \cdots \\ x_1x_2\cdots x_n & = & (-1)^n a_0 \end{cases}$$

Because $x_1, x_2, \cdots, x_n$ are roots of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, the following relation must hold:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_1)(x - x_2)\cdots(x - x_n) = 0$$

Expanding the right ride and comparing the corresponding coefficients yield the desired result.

Practice 2

Let $x_1, x_2, \cdots, x_n$ be the roots of equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ where $a_n \neq 0$. Describe how Vieta's formula, as shown in the previous practice, will need to change?

These relation will still hold if the right side terms are all divided by a_n:

$$\begin{cases} x_1 + x_2 + \cdots + x_n & = & -a_{n-1}/a_n \\ x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots & = & a_{n-2}/a_n \\ \cdots & = & \cdots \\ x_1x_2\cdots x_n & = & (-1)^n a_0/a_n \end{cases}$$

Practice 3

If one root of the equation $x^2 - 6x + m^2 - 2m + 5 = 0$ is 2. Find the value of the other root and m.

let the other root be a, then $a + 2 = 6 \implies a = \boxed{4}$.

It follows that

$$4 \times 2 = m^2 - 2m + 5 \implies m_{1,2} = \boxed{3, -1}$$

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Practice 4

Let x_1 and x_2 be the two roots of equation $x^2 - 3x + 2 = 0$. Find the following values without computing x_1 and x_2 directly.

i) $x_1^4 + x_2^4$

ii) $x_1 - x_2$

Let $y_n = x_1^n + x_2^n$. Then $y_{n+2} - 3y_{n+1} + 2y_n = 0$ or $y_{n+2} = 3y_{n+1} - 2y_n$. Therefore

- $y_0 = x_1^0 + x_2^0 = 2$
- $y_1 = x_1^1 + x_2^1 = 3$ by Vieta's formula
- $y_2 = 3y_1 - 2y_0 = 3 \times 3 - 2 \times 2 = 5$
- $y_3 = 3y_2 - 2y_1 = 3 \times 5 - 2 \times 3 = 9$
- $y_4 = 3y_3 - 2y_2 = 3 \times 9 - 2 \times 5 = \boxed{17}$

In order to compute the value of $(x_1 - x_2)$, let's first compute $(x_1 - x_2)^2$.

$$(x_1 - x_2)^2 = (x_1 + x_2)^{-} 4x_1x_2 = (3)^2 - 4 \times 2 = 1$$

Therefore $x_1 - x_2 = \boxed{\pm 1}$.

Practice 5

Let x_1 and x_2 be the two roots of $x^2 - 3mx + 2(m-1) = 0$. If $\frac{1}{x_1} + \frac{1}{x_2} = \frac{3}{4}$, what is the value of m?

By Vieta's theorem,

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 x_2} = \frac{3m}{2(m-1)} = \frac{3}{4} \implies m = \boxed{-1}$$

Practice 6

If the difference of the two roots of the equation $x^2 + 6x + k = 0$ is 2, what is the value of k?

Let the two roots be x_1 and x_2. Then

$$|x_1 - x_2| = 2 \implies 4 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = 6^2 - 4k \implies k = 8$$

Practice 7

If the two roots of $x^2 + ax + b + 1 = 0$ are positive integers, show that $a^2 + b^2$ cannot be a prime number.

Let the two roots be x_1 and x_2. By Vieta's theorem, we have

$$x_1 + x_2 = -a \text{ and } x_1x_2 = (b+1)$$

Then

$$a^2 + b^2 = (x_1 + x_2)^2 + (x_1x_2 - 1)^2 = x_1^2 + x_2^2 + x_1^2x_2^2 + 1 = (x_1^2 + 1)(x_2^2 + 1)$$

Because both x_1 and x_2 are positive integers, both $(x_1^2 + 1)$ and $(x_2^2 + 1)$ are positive integers greater than 1. Hence $(a^2 + b^2)$ have two positive divisors both of which are greater than 1. This means it is not prime.

Practice 8

Find the sum of all possible integer values of a such that the equation

$$(a+1)x^2 - (a^2+1)x + (2a^2 - 6) = 0$$

is solvable in integers.

We need to ensure an equation is quadratic before applying Vieta's theorem.

If $(a+1) = 0$ or $a = -1$, this equation has an integer root of -2.

When $a + 1 \neq 0$, this equation is quadratic. Suppose its two roots are x_1 and x_2. Then by Vieta's formula, we have

$$x_1 + x_2 = \frac{a^2+1}{a+1} = (a-1) + \frac{2}{a+1}$$

If both x_1 and x_2 are integers, $(a+1)$ must be divisible by 2. This implies that $a = 1, 0, -2$ or -3.

- $a = 1 \implies 2x^2 - 2x - 4 = 0 \implies x_{1,2} = 2, -1$

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- $a = 0 \implies x^2 - x - 6 = 0 \implies x_{1,2} = 3, -2$
- $a = -2 \implies -x^2 - 5x + 2 = 0$, no integer solution
- $a = -3 \implies -2x^2 - 10x + 12 = 0 \implies x_{1,2} = 1, -6$

Therefore the answer is $-1 + 1 + 0 - 3 = \boxed{-3}$.

Practice 9

Let x_1 and x_2 be the two real roots of the equation $x^2-2(k+1)x+k^2+2 = 0$. If $(x_1+1)(x_2+1) = 8$, find the value of k

By Vieta's theorem: $x_1 + x_2 = 2(k + 1)$ and $x_1x_2 = k^2 + 2$. Therefore

$$8 = (x_1 + 1)(x_2 + 1) = x_1x_2 + (x_1 + x_2) + 1 = k^2 + 2 + 2(k + 1) + 1$$

or

$$k^2 + 2k - 3 - 0 \implies k_{1,2} = -3, 1$$

Because the original equation has two real roots, we must ensure **its determinant is positive**. This will eliminate $k = -3$. Hence the final answer is $k = \boxed{1}$.

Practice 10

Three of the roots of $x^4 + ax^2 + bx + c = 0$ are 2,-3, and 5. Find the value of $a + b + c$.
(Ref 1998 HMMT)

The coefficient of x^3 is 0, so the sum of all the four roots is 0. This implies the fourth root must be -4.

It is then possible to apply Vieta's formula to compute a, b, and c before summing them up. However there is a quicker way to get the result. Because the four roots are 2,-3,5, and -4, therefore the following relation must hold:

$$x^4 + ax^2 + bx + c = (x - 2)(x + 3)(x - 5)(x + 4)$$

This is an identity which will hold for any value of x. Let's set $x = 1$:

$$1 + a + b + c = (1 - 2)(1 + 3)(1 - 5)(1 + 4) = 80 \implies a + b + c = \boxed{79}$$

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Practice 11

If real number m and n satisfy $mn \neq 1$ and $19m^2 + 99m + 1 = 0$ and $19 + 99n + n^2 = 0$, what is the value of $\frac{mn+4m+1}{n}$?

It is apparent that $n \neq 0$ and $m \neq \frac{1}{n}$. Dividing both sides of the 2^{nd} equation by n^2 leads to

$$19\left(\frac{1}{n}\right)^2 + 99\left(\frac{1}{n}\right) + 1 = 0$$

Therefore m and $\frac{1}{n}$ are two distinct roots of the following quadratic equation:

$$19t^2 + 99t + 1 = 0$$

It follows that

$$\frac{mn + 4m + 1}{n} = m + \left(\frac{1}{n}\right) + 4m\left(\frac{1}{n}\right) = \left(-\frac{99}{19}\right) + 4 \times \left(\frac{1}{19}\right) = \boxed{-5}$$