

---

## Practice

---

### Vieta's Formula (Basic)

---



Math for Gifted Students

<http://www.mathallstar.org>

Practice

## Vieta's Formula (Basic)



## Practice 1

Let  $x_1, x_2, \dots, x_n$  be the roots of equation  $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$ . Show that

$$\left\{ \begin{array}{lcl} x_1 + x_2 + \dots + x_n & = & -a_{n-1} \\ x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots & = & a_{n-2} \\ \dots & = & \dots \\ x_1x_2 \dots x_n & = & (-1)^n a_0 \end{array} \right.$$

## Practice 2

Let  $x_1, x_2, \dots, x_n$  be the roots of equation  $a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$  where  $a_n \neq 0$ . Describe how Vieta's formula, as shown in the previous practice, will need to change?

## Practice 3

If one root of the equation  $x^2 - 6x + m^2 - 2m + 5 = 0$  is 2. Find the value of the other root and  $m$ .

## Practice 4

Let  $x_1$  and  $x_2$  be the two roots of equation  $x^2 - 3x + 2 = 0$ . Find the following values without computing  $x_1$  and  $x_2$  directly.

- i)  $x_1^4 + x_2^4$
- ii)  $x_1 - x_2$

## Practice 5

Let  $x_1$  and  $x_2$  be the two roots of  $x^2 - 3mx + 2(m-1) = 0$ . If  $\frac{1}{x_1} + \frac{1}{x_2} = \frac{3}{4}$ , what is the value of  $m$ ?

## Practice 6

If the difference of the two roots of the equation  $x^2 + 6x + k = 0$  is 2, what is the value of  $k$ ?

Practice

## Vieta's Formula (Basic)



## Practice 7

If the two roots of  $x^2 + ax + b + 1 = 0$  are positive integers, show that  $a^2 + b^2$  cannot be a prime number.

## Practice 8

Find the sum of all possible integer values of  $a$  such that the equation

$$(a + 1)x^2 - (a^2 + 1)x + (2a^2 - 6) = 0$$

is solvable in integers.

## Practice 9

Let  $x_1$  and  $x_2$  be the two real roots of the equation  $x^2 - 2(k+1)x + k^2 + 2 = 0$ . If  $(x_1+1)(x_2+1) = 8$ , find the value of  $k$

## Practice 10

Three of the roots of  $x^4 + ax^2 + bx + c = 0$  are 2, -3, and 5. Find the value of  $a + b + c$ .

(Ref 1998 HMMT)

## Practice 11

If real number  $m$  and  $n$  satisfy  $mn \neq 1$  and  $19m^2 + 99m + 1 = 0$  and  $19 + 99n + n^2 = 0$ , what is the value of  $\frac{mn+4m+1}{n}$ ?

---

## Reference Solutions

---

Practice

## Vieta's Formula (Basic)



## Practice 1

Let  $x_1, x_2, \dots, x_n$  be the roots of equation  $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$ . Show that

$$\left\{ \begin{array}{rcl} x_1 + x_2 + \dots + x_n & = & -a_{n-1} \\ x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots & = & a_{n-2} \\ \dots & = & \dots \\ x_1x_2 \dots x_n & = & (-1)^n a_0 \end{array} \right.$$

Because  $x_1, x_2, \dots, x_n$  are roots of  $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$ , the following relation must hold:

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = (x - x_1)(x - x_2) \dots (x - x_n) = 0$$

Expanding the right side and comparing the corresponding coefficients yield the desired result.

## Practice 2

Let  $x_1, x_2, \dots, x_n$  be the roots of equation  $a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$  where  $a_n \neq 0$ . Describe how Vieta's formula, as shown in the previous practice, will need to change?

These relation will still hold if the right side terms are all divided by  $a_n$ :

$$\left\{ \begin{array}{rcl} x_1 + x_2 + \dots + x_n & = & -a_{n-1}/a_n \\ x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots & = & a_{n-2}/a_n \\ \dots & = & \dots \\ x_1x_2 \dots x_n & = & (-1)^n a_0/a_n \end{array} \right.$$

## Practice 3

If one root of the equation  $x^2 - 6x + m^2 - 2m + 5 = 0$  is 2. Find the value of the other root and  $m$ .

let the other root be  $a$ , then  $a + 2 = 6 \implies a = \boxed{4}$ .

It follows that

$$4 \times 2 = m^2 - 2m + 5 \implies m_{1,2} = \boxed{3, -1}$$

Practice

## Vieta's Formula (Basic)



## Practice 4

Let  $x_1$  and  $x_2$  be the two roots of equation  $x^2 - 3x + 2 = 0$ . Find the following values without computing  $x_1$  and  $x_2$  directly.

- i)  $x_1^4 + x_2^4$
- ii)  $x_1 - x_2$

Let  $y_n = x_1^n + x_2^n$ . Then  $y_{n+2} - 3y_{n+1} + 2y_n = 0$  or  $y_{n+2} = 3y_{n+1} - 2y_n$ . Therefore

- $y_0 = x_1^0 + x_2^0 = 2$
- $y_1 = x_1^1 + x_2^1 = 3$  by Vieta's formula
- $y_2 = 3y_1 - 2y_0 = 3 \times 3 - 2 \times 2 = 5$
- $y_3 = 3y_2 - 2y_1 = 3 \times 5 - 2 \times 3 = 9$
- $y_4 = 3y_3 - 2y_2 = 3 \times 9 - 2 \times 5 = \boxed{17}$

In order to compute the value of  $(x_1 - x_2)$ , let's first compute  $(x_1 - x_2)^2$ .

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = (3)^2 - 4 \times 2 = 1$$

Therefore  $x_1 - x_2 = \boxed{\pm 1}$ .

## Practice 5

Let  $x_1$  and  $x_2$  be the two roots of  $x^2 - 3mx + 2(m-1) = 0$ . If  $\frac{1}{x_1} + \frac{1}{x_2} = \frac{3}{4}$ , what is the value of  $m$ ?

By Vieta's theorem,

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 x_2} = \frac{3m}{2(m-1)} = \frac{3}{4} \implies m = \boxed{-1}$$

## Practice 6

If the difference of the two roots of the equation  $x^2 + 6x + k = 0$  is 2, what is the value of  $k$ ?

Practice

## Vieta's Formula (Basic)



Let the two roots be  $x_1$  and  $x_2$ . Then

$$|x_1 - x_2| = 2 \implies 4 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = 6^2 - 4k \implies k = 8$$

**Practice 7**

If the two roots of  $x^2 + ax + b + 1 = 0$  are positive integers, show that  $a^2 + b^2$  cannot be a prime number.

Let the two roots be  $x_1$  and  $x_2$ . By Vieta's theorem, we have

$$x_1 + x_2 = -a \text{ and } x_1x_2 = (b + 1)$$

Then

$$a^2 + b^2 = (x_1 + x_2)^2 + (x_1x_2 - 1)^2 = x_1^2 + x_2^2 + x_1^2x_2^2 + 1 = (x_1^2 + 1)(x_2^2 + 1)$$

Because both  $x_1$  and  $x_2$  are positive integers, both  $(x_1^2 + 1)$  and  $(x_2^2 + 1)$  are positive integers greater than 1. Hence  $(a^2 + b^2)$  have two positive divisors both of which are greater than 1. This means it is not prime.

**Practice 8**

Find the sum of all possible integer values of  $a$  such that the equation

$$(a + 1)x^2 - (a^2 + 1)x + (2a^2 - 6) = 0$$

is solvable in integers.

**We need to ensure an equation is quadratic before applying Vieta's theorem.**

If  $(a + 1) = 0$  or  $a = -1$ , this equation has an integer root of  $-2$ .

When  $a + 1 \neq 0$ , this equation is quadratic. Suppose its two roots are  $x_1$  and  $x_2$ . Then by Vieta's formula, we have

$$x_1 + x_2 = \frac{a^2 + 1}{a + 1} = (a - 1) + \frac{2}{a + 1}$$

If both  $x_1$  and  $x_2$  are integers,  $(a + 1)$  must be divisible by 2. This implies that  $a = 1, 0, -2$  or  $-3$ .

- $a = 1 \implies 2x^2 - 2x - 4 = 0 \implies x_{1,2} = 2, -1$

Practice

## Vieta's Formula (Basic)



- $a = 0 \implies x^2 - x - 6 = 0 \implies x_{1,2} = 3, -2$
- $a = -2 \implies -x^2 - 5x + 2 = 0$ , no integer solution
- $a = -3 \implies -2x^2 - 10x + 12 = 0 \implies x_{1,2} = 1, -6$

Therefore the answer is  $-1 + 1 + 0 - 3 = \boxed{-3}$ .

— Practice 9 —

Let  $x_1$  and  $x_2$  be the two real roots of the equation  $x^2 - 2(k+1)x + k^2 + 2 = 0$ . If  $(x_1+1)(x_2+1) = 8$ , find the value of  $k$

By Vieta's theorem:  $x_1 + x_2 = 2(k + 1)$  and  $x_1x_2 = k^2 + 2$ . Therefore

$$8 = (x_1 + 1)(x_2 + 1) = x_1x_2 + (x_1 + x_2) + 1 = k^2 + 2 + 2(k + 1) + 1$$

or

$$k^2 + 2k - 3 - 0 \implies k_{1,2} = -3, 1$$

Because the original equation has two real roots, we must ensure **its determinant is positive**. This will eliminate  $k = -3$ . Hence the final answer is  $k = \boxed{1}$ .

— Practice 10 —

Three of the roots of  $x^4 + ax^2 + bx + c = 0$  are 2, -3, and 5. Find the value of  $a + b + c$ .

(Ref 1998 HMMT)

The coefficient of  $x^3$  is 0, so the sum of all the four roots is 0. This implies the fourth root must be -4.

It is then possible to apply Vieta's formula to compute  $a$ ,  $b$ , and  $c$  before summing them up. However there is a quicker way to get the result. Because the four roots are 2, -3, 5, and -4, therefore the following relation must hold:

$$x^4 + ax^2 + bx + c = (x - 2)(x + 3)(x - 5)(x + 4)$$

This is an identity which will hold for any value of  $x$ . Let's set  $x = 1$ :

$$1 + a + b + c = (1 - 2)(1 + 3)(1 - 5)(1 + 4) = 80 \implies a + b + c = \boxed{79}$$

Practice

## Vieta's Formula (Basic)



## Practice 11

If real number  $m$  and  $n$  satisfy  $mn \neq 1$  and  $19m^2 + 99m + 1 = 0$  and  $19 + 99n + n^2 = 0$ , what is the value of  $\frac{mn+4m+1}{n}$ ?

It is apparent that  $n \neq 0$  and  $m \neq \frac{1}{n}$ . Dividing both sides of the  $2^{nd}$  equation by  $n^2$  leads to

$$19\left(\frac{1}{n}\right)^2 + 99\left(\frac{1}{n}\right) + 1 = 0$$

Therefore  $m$  and  $\frac{1}{n}$  are two distinct roots of the following quadratic equation:

$$19t^2 + 99t + 1 = 0$$

It follows that

$$\frac{mn + 4m + 1}{n} = m + \left(\frac{1}{n}\right) + 4m\left(\frac{1}{n}\right) = \left(-\frac{99}{19}\right) + 4 \times \left(\frac{1}{19}\right) = \boxed{-5}$$